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# A Note on a Parameter Depending Datko Theorem Applied to Stochastic Systems*

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## 0 Introduction

Let us denote by $y_\lambda(\cdot, s, x)$, for all $\lambda$ belonging to a set $\Lambda$ of parameters, the solution of a stochastic linear differential equation as:

$$\begin{cases} d_t y_\lambda(t,s,x) = A_\lambda y_\lambda(t,s,x)dt + C_\lambda y_\lambda(t,s,x)dW_t \\ y_\lambda(s,s,x) = x \end{cases} \tag{0.1}$$

where: $W$ is a Wiener process (finite- or infinite-dimensional) defined on the stochastic basis $(\Omega, \mathcal{E}, \mathcal{F}_t, \mathbb{P})$ and adapted to $\mathcal{F}$; $A_\lambda$ and $C_\lambda$ are linear operators on an Hilbert space $H$; $x$ belongs to $\mathbf{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}, H)$.
By a generalized version of the Datko Theorem (see [2] and [5]) we know that, for a fixed $\lambda \in \Lambda$, the two following statements are equivalent:

*i)* $\mathbb{E}\int_0^\infty \|y_\lambda(t,s,x)\|^2\, dt \le c_\lambda \mathbb{E}\|x\|^2$ for some $c_\lambda \in \mathbb{R}$, for all $t \ge s \ge 0$ and for all $x \in \mathbf{L}^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$

*ii)* $\exists M_\lambda \ge 0$, $a_\lambda > 0$ s.t. $\mathbb{E}\|y_\lambda(t,s,x)\|^2 \le M_\lambda^2 e^{-2a_\lambda(t-s)} \mathbb{E}\|x\|^2$ for all $t \ge s \ge 0$ and all $x \in \mathbf{L}^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$.

Recently (see [8]) it has been proved, in the deterministic case ($C = 0$ in equation (0.1)), that: if *i)* holds for all $\lambda \in \Lambda$ and uniformly in $\lambda$ (that is with $c_\lambda = c$), then *ii)* is verified for all $\lambda \in \Lambda$ with $M_\lambda = M$ and $a_\lambda = a > 0$.

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Here we show that a similar result holds for a family of stochastic systems of the kind of (0.1). The proof is based (as in [5] and [8]) on the semigroup properties of the family of linear operators $T_\lambda(t,s) : \mathbf{L}^2(\Omega, \mathcal{F}_s, \mathbb{P}, H) \to \mathbf{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}, H)$ defined by $T_\lambda(t,s)x = y_\lambda(t,s,x)$. Notice that in the deterministic case the above spaces coincide (being both equal to $H$); this is the main difference between the situation arising from deterministic equations and the one arising from stochastic equations and is the reason why our proof can not follow [8] in straight-forward way.

The interest on parameter depending stochastic differential equations (SDEs) arises in a very natural way, for instance, in ergodic control and in adaptive control of stochastic systems (see [7]) and [3]). In that same framework it is sometimes useful to know that a parameterized class of SDEs have solutions that decay exponentially to zero uniformly on the parameter. Indeed in §2 we exploit the general result, proved in §1, to obtain the uniform decay of the optimal states of a class of linear, infinite dimensional, stochastic systems when a *uniform detectability* assumption holds. Finally (see Example 2.1) we show that, for a particular parameter depending controlled stochastic system coming from ergodic control of affine stochastic partial differential equations (see [7]), the above mentioned uniform detectability condition is verified. Such an example of application was, in fact, our starting motivation.

## 1 Main Result

Let $\mathcal{Z}$ be a Banach space (norm $|\cdot|$) and let, for all $t \geq s \geq 0$, and for all $\lambda \in \Lambda$ ($\Lambda$ being a fixed set of parameters), $T_\lambda(t,s)$ be a linear operator with domain $\mathcal{Y}_s \subset \mathcal{Z}$. Assume that the family $\{T_\lambda(t,s) : \lambda \in \Lambda; \quad t \geq s \geq 0\}$ verifies the following conditions ($\forall t \geq s \geq 0$, $\forall \lambda \in \Lambda$ and for all $x \in \mathcal{Y}_s$):

$$T_\lambda(\cdot, s)x \text{ is a continuous map } [s, \infty[\to \mathcal{Z} \tag{1.1}$$

$$T_\lambda(t,s)\mathcal{Y}_s \subset \mathcal{Y}_t \tag{1.2}$$

$$T_\lambda(s,s)x = x \text{ and } \quad T_\lambda(t,s)x = T_\lambda(t,\tau)T_\lambda(\tau,s)x, \;\; \forall \tau \in [s,t] \tag{1.3}$$

$$|T_\lambda(t,s)x| \leq Re^{\rho(t-s)}|x|, \quad \text{for some } R \geq 0,\, \rho \in \mathbb{R} \tag{1.4}$$

$$\int_s^\infty |T_\lambda(t,s)x|^p \, dt \leq c_1 |x|^p, \quad \text{for some } c_1 \geq 0,\, p \geq 1 \tag{1.5}$$

then the following holds:

**Lemma 1.1** *There exist $c_2, c_3 \in \mathbb{R}$ such that, $\forall t \geq s \geq 0$, $\forall \lambda \in \Lambda$, $\forall x \in \mathcal{Y}_s$:*

$$|T_\lambda(t,s)x| \leq c_2|x| \tag{1.6}$$

$$|T_\lambda(t,s)x| \leq c_3(t-s)^{-1/p}|x|. \tag{1.7}$$

**Proof:** By (1.3) and (1.4) we obtain that for all $t \geq \tau \geq s \geq 0$, $\forall \lambda \in \Lambda$ and $\forall x \in \mathcal{Y}_s$:

$$|T_\lambda(t,s)x|^p \leq R^p e^{p\rho(t-\tau)}|T_\lambda(\tau,s)x|^p.$$

Now, for all $t \geq s+1$, we have:

$$\begin{aligned} |T_\lambda(t,s)x|^p &= \int_{t-1}^t |T_\lambda(t,\tau)T_\lambda(\tau,s)x|^p\,d\tau \leq \\ &\leq R^p e^{p\rho} \int_{t-1}^t |T_\lambda(\tau,s)x|^p\,d\tau \leq R^p e^{p\rho} c_1 |x|^p \end{aligned}$$

the last inequality following from (1.5). The above relation combined with assumption (1.4) immediately yields (1.6). Moreover:

$$\begin{aligned} (t-s)\,|T_\lambda(t,s)x|^p = \int_s^t |T_\lambda(t,\tau)T_\lambda(\tau,s)x|^p\,d\tau \leq \\ \leq c_2^p \int_s^t |T_\lambda(\tau,s)x|^p\,d\tau \leq c_2^p c_1 |x|^p \end{aligned}$$

so (1.7) follows letting $c_3 = c_2 c_1^{1/p}$. □

We can now conclude our argument.

**Proposition 1.2** *If assumptions (1.1) - (1.5) hold then there exist $\mathcal{M} \geq 0$ and $\alpha > 0$ such that, for all $t \geq s \geq 0$, $\lambda \in \Lambda$, $x \in \mathcal{Y}_s$:*

$$|T_\lambda(t,s)x| \leq \mathcal{M}e^{-\alpha(t-s)}|x|. \tag{1.8}$$

**Proof:** Let $L = 2^p c_3^p$, by (1.7) we get that, $\forall x \in \mathcal{Y}_s$, $\forall \lambda \in \Lambda$:

$$|T_\lambda(t,s)x| \leq \frac{1}{2}|x| \quad \text{whenever} \quad t-s \geq L.$$

So if $nL \leq (t-s) < (n+1)L$ for some $n \in \mathbb{N}$, applying $n$ times the above relation, we obtain:

$$|T_\lambda(t,s)x| \leq c_2 2^{-n}|x| \leq 2c_2 2^{-(t-s)L^{-1}}|x| \leq \mathcal{M}_0 e^{-\alpha(t-s)}|x| \tag{1.9}$$

where $\alpha = L^{-1}\log(2) > 0$, $\mathcal{M}_0 = 2c_2$ (for the last inequality see also [5]). Combining inequality (1.9) (holding whenever $t-s \geq L$) and assumption (1.4) we can conclude that, letting $\mathcal{M} = \mathcal{M}_0 \vee c_2$, our claim holds. □

**Remark 1.3** The above argument follows [8] in the proof of Lemma 1.1 except that we avoid using the uniform boundedness theorem (since it is not clear on which space it can be applied). The conclusion then follows [5].

Finally in [8] it is shown that assumption (1.4) can not be avoided.

**Example 1.1** *Let (as in §0) $H$ be a separable Hilbert space (norm $\|\cdot\|$, product $\langle\langle\cdot,\cdot\rangle\rangle$) and $(\Omega, \mathcal{E}, \mathcal{F}_t, \mathbb{P})$ be a standard stochastic base ($\{\mathcal{F}_t : t \geq 0\}$ being a filtration in $\mathcal{E}$). Moreover define:*

$$\mathcal{Z} = \mathbf{L}^2(\Omega, \mathcal{E}, \mathbb{P}, H) \quad \text{and} \quad \mathcal{Y}_s = \mathbf{L}^2(\Omega, \mathcal{F}_s, \mathbb{P}, H) \ \ (\forall s \geq 0)$$

*and, for all $\lambda \in \Lambda$, $T_\lambda(t,s)x = y_\lambda(\cdot, s, x)$ (where $y_\lambda(\cdot, s, x)$ is the solution of (0.1)).*
*Under very general assumptions on the coefficients in (0.1) (see the next section or [6]) $T_\lambda(t,s)$ is a well defined bounded linear operator from $\mathcal{Y}_s$ into $\mathcal{Y}_t$, moreover (1.1), (1.2) and (1.3) hold.*
*Therefore if, for all $t \geq s \geq 0$ and all $x \in \mathbf{L}^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$:*

$$\mathbb{E}\|y_\lambda(t,s,x)\|^2 \leq Re^{(t-s)\varrho}\mathbb{E}\|x\|^2 \quad \textit{for some } R \geq 0,\ \varrho \in \mathbb{R} \tag{1.10}$$

$$\int_s^\infty \mathbb{E}\|y_\lambda(\sigma,s,x)\|^2 \, d\sigma \leq c \qquad \textit{for some } c > 0, \tag{1.11}$$

*then by Proposition 1.2 there exists $M \geq 0$ and $a > 0$ such that:*

$$\mathbb{E}\|y_\lambda(t,s,x)\|^2 \leq M^2 e^{-2a(t-s)}\mathbb{E}\|x\|^2 .$$

## 2 Uniform Detectability

We want to apply the above result to deduce the uniform exponential decay of the optimal states of a class of detectable stochastic linear quadratic control problems. Let us consider the following "state equation":

$$\begin{cases} d_t y(t) = (Ay + Bu)dt + Cyd\beta_t \\ y(0) = x \end{cases} \tag{2.1}$$

and the following "infinite horizon cost functional":

$$J(x,u) = \mathbb{E}\int_0^\infty \left( \left\|\sqrt{S}y(s)\right\|^2 + \|u(s)\|^2 \right) ds \tag{2.2}$$

where $y$ represents the state of the dynamic system described by (2.1) and $u$ is the control introduced in it.

Let us specify some assumptions and notations (for simplicity we consider a one-dimensional noise):

- $H$ and $(\Omega, \mathcal{E}, \mathcal{F}, \mathbb{P})$ are as in Example 1.1; $\beta$ is a one-dimensional brownian motion defined on $(\Omega, \mathcal{E}, \mathcal{F}, \mathbb{P})$ and adapted to $\mathcal{F}$.
- $A : \mathcal{D}(A) \subset H \to H$ is a regularly dissipative operator. That is, we recall, there exists an hilbert space $V \subset H$ ($V$ is endowed with norm $|\cdot|$) with continuous and dense inclusion. Moreover it is defined in $V$ a continuous bilinear form $a(\cdot,\cdot)$ verifying $-a(v,v) \geq c|v|^2 - \ell\|v\|^2$ for some $c > 0$, $\ell \geq 0$ and all $v \in V$. Then $A$ is defined as follows:

  $$\begin{array}{l} \mathcal{D}(A) = \{x \in V : \text{ the map } y \to a(x,y) \text{ is continuous in } H\} \\ \forall x \in \mathcal{D}(A) \;\; Ax \text{ verifies } a(x,y) = \langle\langle Ax, y\rangle\rangle \;\forall y \in V. \end{array}$$

  We also recall that such an $A$ generates an analytic semigroup of pseudo-contractions.
- $C \in \mathcal{L}(V,H)$, and the following "ellipticity" condition holds:

  $$\|Cx\|^2 \leq -2\eta a(x,x) + \chi\,\|x\|^2 \qquad \text{for some } \eta \in ]0,1[,\ \chi \in \mathbb{R}.$$
- $B \in \mathcal{L}(H)$, $S \in \mathcal{L}(H)$ with $S$ self adjoint non negative.
- If $K$ is an Hilbert space by $\mathbf{M}^2_{\mathcal{P}}(s,T,K)$ (resp. $\mathbf{M}^2_{\mathcal{P}}(s,\infty,K)$) we denote the closed subspace of $\mathbf{L}^2(\Omega \times [s,T], \mathcal{E} \otimes \mathcal{B}([s,T]), \mathbb{P} \otimes \mu, K)$ (resp. of $\mathbf{L}^2(\Omega \times [s,+\infty), \mathcal{E} \otimes \mathcal{B}([s,+\infty), \mathbb{P} \otimes \mu, K)$ ) (where $\mu$ is the Lebesgue's measure and $\mathcal{B}$ denotes the standard Borel $\sigma$-field) given by all equivalence classes that contain a predictable process with respect to the filtration $\mathcal{F}$.
- $x$ belongs to $\mathbf{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}, H)$.

Under this assumption we can show (for all fixed $T > 0$) the following existence and uniqueness result (see [6]):

**Proposition 2.1** *For all $u \in \mathbf{M}^2_{\mathcal{P}}(0,T,H)$, there exists a unique mild solution $y \in \mathbf{M}^2_{\mathcal{P}}(0,T,V)$ of (2.1). Moreover the map $t \to y(t)$, considered as a map with values in $\mathbf{L}^2(\Omega, \mathcal{E}, \mathbb{P}, H)$), is continuous.*

We give here the following definitions:

**Definition 2.1** *We say that $(A,B,C)$ is stabilizable relatively to $\sqrt{S}$ if, for all $x \in \mathbf{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}, H)$, $\exists u \in \mathbf{M}^2_{\mathcal{P}}(0,\infty,H)$ such that $J(x,u) < \infty$.*

**Definition 2.2** *We say that $(A, \sqrt{S}, C)$ is detectable if there exists $Q \in \mathcal{L}(H)$, $M \geq 0$ and $a > 0$ such that, letting, for all $s \geq 0$ and for all $x \in \mathbf{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}, H)$, $V(\cdot,s)x$ be the solution of:*

$$\left\{ \begin{array}{l} d_t V(t,s)x = (A - Q\sqrt{S})V(t,s)x dt + CV(t,s)x d\beta_t \\ V(s,s)x = x \end{array} \right. \tag{2.3}$$

*then* $\mathbb{E}\|V(t,s)x\|^2 \leq M^2 e^{-2a(t-s)}\mathbb{E}\|x\|^2$ *for all* $t \geq s$[†] *(the existence and uniqueness of the solution of equation (2.3) is proved exactly as Proposition 2.1).*

**Remark 2.2** In [6] it is shown that if $(A,B,C)$ is stabilizable relatively to $\sqrt{S}$, then there exists $X \in \mathcal{L}(H)$ self-adjoint and non negative such that:

$$\inf_{u \in \mathbf{M}^2_{\mathcal{P}}(0,\infty,H)} J(x,u) = \mathbb{E}\langle\langle Xx, x\rangle\rangle.$$

Moreover if for all $s \geq 0$ and all $x \in \mathbf{L}^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$ we denote by $\xi(\cdot, s, x)$ the solution of the following "closed loop equation":

$$\left\{ \begin{array}{l} d_t\xi(t,s,x) = (A - BB^*X)\xi(t,s,x)dt + C\xi(t,s,x)d\beta_t \\ \xi(s,s,x) = x \end{array} \right. \tag{2.4}$$

then $\xi(\cdot,s,x) \in \mathbf{M}^2_{\mathcal{P}}(s,\infty,H)$ and:

$$\begin{aligned} \inf_{u \in \mathbf{M}^2_{\mathcal{P}}(0,\infty,H)} J(x,u) &= J(x, -B^*X\xi(\cdot,0,x)) = \\ &= \mathbb{E}\int_0^\infty \left(\left\|\sqrt{S}\xi(t,0,x)\right\|^2 + \|B^*X\xi(t,0,x)\|^2\right) dt \end{aligned}$$

In other words, $\xi(\cdot,0,x)$ is the optimal state and $-B^*X\xi(\cdot,0,x)$ is the optimal control corresponding to our control problem. □

Let us now fix a set $\Lambda$ and introduce the following class of control problems:

$$\left\{ \begin{array}{l} d_t y_\lambda(t) = (A_\lambda y_\lambda + B_\lambda u)dt + C_\lambda y_\lambda d\beta_t \\ y_\lambda(0) = x \end{array} \right. \tag{2.5}$$

$$J_\lambda(x,u) = \mathbb{E}\int_0^\infty \left(\left\|\sqrt{S_\lambda} y_\lambda(s)\right\|^2 + \|u(s)\|^2\right) ds \tag{2.6}$$

where $H$ and $\beta$ are defined as above and $A_\lambda$, $B_\lambda$, $C_\lambda$, $S_\lambda$ verify, for all $\lambda \in \Lambda$, the previous assumptions. Moreover we suppose that the following hypotheses hold:

**Hypothesis 2.1** *The operators* $B_\lambda$ *are bounded uniformly in* $\lambda \in \Lambda$ *(that is* $\sup_\lambda \|B_\lambda\|_{\mathcal{L}(H)} < \infty$*).*

**Hypothesis 2.2** *For all* $\lambda \in \Lambda$ $(A_\lambda, B_\lambda, C_\lambda)$ *is stabilizable. Moreover there exists* $\gamma > 0$ *such that, for all* $\lambda \in \Lambda$*:*

$$\inf_{u \in \mathbf{M}^2_{\mathcal{P}}(0,\infty,H)} J_\lambda(x,u) \leq \gamma\mathbb{E}\|x\|^2.$$

[†]Notice that, by the Datko Theorem applied to stochastic systems (see [5]), this is equivalent to: $\sup_{x,s} \mathbb{E}\int_s^\infty \|V(t,s)x\|^2\, dt < \infty$

**Hypothesis 2.3** *For all $\lambda \in \Lambda$, $(A_\lambda, \sqrt{S_\lambda}, C_\lambda)$ is detectable.*
*Moreover the operators $Q_\lambda$ ($Q_\lambda$ is the operator guaranteed by definition 2.2 and corresponding to the control problem with index $\lambda$) can be chosen so that:* $\sup_{\lambda\in\Lambda} \|Q_\lambda\|_{\mathcal{L}(H)} < +\infty$.
*Finally, defining, for all $\lambda \in \Lambda$, $V_\lambda$ as in (2.3), there exists $U \geq 0$ and $b > 0$ such that $\mathbb{E}\|V_\lambda(t,s)x\|^2 \leq U^2 e^{-2b(t-s)} \|x\|^2$ for all $t \geq s$, all $\lambda \in \Lambda$ and all $x \in \mathbf{L}^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$.*‡

Now let $\mathcal{Z}$ and $\mathcal{Y}_s$ as in Example 1.1. Define, $\forall\lambda \in \Lambda$, $\xi_\lambda$ as in (2.4) (corresponding to the control problem with index $\lambda$). In order to apply the result of Example 1.1 to the processes $\xi_\lambda(\cdot, s, x)$, obtaining their uniform exponential decay, we need to show that conditions (1.10) and (1.11) hold:

**Lemma 2.3** *If hypotheses 2.1, 2.2, 2.3 hold then $\{\xi_\lambda(\cdot,\cdot,\cdot) : \lambda \in \Lambda\}$ verify conditions (1.10) and (1.11).*

**Proof:** The proof follows [1]. We can write:

$$\begin{cases} d_t\xi_\lambda(t,s,x) = (A_\lambda - Q_\lambda\sqrt{S_\lambda})\xi_\lambda(t,s,x)dt + \varphi_\lambda(t,s,x)dt + \\ \qquad\qquad\qquad\qquad + C_\lambda\xi_\lambda(t,s,x)d\beta_t \\ \xi_\lambda(s,s,x) = x \end{cases}$$

where $\varphi_\lambda(t,s,x) = (Q_\lambda\sqrt{S_\lambda} + B_\lambda B_\lambda^* X_\lambda)\xi_\lambda(t,s,x)$. So $\xi_\lambda(t,s,x)$ can be represented, by a standard variation of constants formula as:

$$\xi_\lambda(t,s,x) = V_\lambda(t,s)x + \int_0^t V_\lambda(t,\sigma)\varphi_\lambda(\sigma,s,x)d\sigma. \tag{2.7}$$

Notice that by hypotheses 2.1, 2.2 and remark 2.2 it follows that there exists $\delta$ such that, for all $s \geq 0$, $\lambda \in \Lambda$, $x \in \mathbf{L}^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$:

$$\mathbb{E}\int_s^\infty \|\varphi_\lambda(t,s,x)\|^2 dt \leq \delta\mathbb{E}\|x\|^2. \tag{2.8}$$

Let us now come back to relation (2.7). By hypothesis 2.3 we have:

$$\begin{aligned} &\mathbb{E}\|\xi_\lambda(t,s,x)\|^2 \leq \\ &\leq 2U^2\left(e^{-2b(t-s)}\mathbb{E}\|x\|^2 + \left(\int_s^t e^{-b(\sigma-s)}\left(\mathbb{E}\|\varphi_\lambda(\sigma,s,x)\|^2\right)^{1/2} d\sigma\right)^2\right) \\ &\leq 2U^2\left(e^{-b(t-s)}\mathbb{E}\|x\|^2 + b^{-1}\int_s^t e^{-b(\sigma-s)}\mathbb{E}\|\varphi_\lambda(\sigma,s,x)\|^2 d\sigma\right) \end{aligned}$$

‡The above assumption has been exposed in this way for simplicity's sake, but it can be expressed in a slightly weaker form using the results included in the previous section

so condition (1.4) follows from (2.8).

Finally integrating the above relation we have (again by (2.8)):

$$\int_s^{\infty} \mathbb{E}\|\xi_\lambda(t,s,x)\|^2\,dt \le U^2\left(b^{-1}+2b^{-2}\delta\right)\mathbb{E}\|x\|^2$$

that is exactly what we wanted to show. □

So by Proposition 1.2 we can conclude:

**Corollary 2.4** *If hypotheses 2.1, 2.2, 2.3 are verified then there exist $M \ge 0$ and $a > 0$ such that $\forall t \ge s \ge 0$, $\forall \lambda \in \Lambda$, $\forall x \in \mathbf{L}^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$:*

$$\mathbb{E}\left\|\xi_\lambda(t,s,x)\right\|^2 \le M^2 e^{-(t-s)a}\mathbb{E}\|x\|^2.$$

**Example 2.1** Let $\Lambda = \mathbb{R}^+$, $A_\lambda = A - \lambda I$, $B_\lambda = B$, $C_\lambda = C$, $S_\lambda = S$ and assume that $(A, B, C)$ is $\sqrt{S}$-stabilizable and $(A, \sqrt{S}, C)$ is detectable.
If $u_\lambda(t) = e^{-\lambda t}u(t)$ and $y$ is the solution of (2.1) then the solution $y_\lambda$ of:

$$\begin{cases} d_t y_\lambda(t) = (A_\lambda y_\lambda + B_\lambda u_\lambda)dt + C y_\lambda d\beta_t \\ y_\lambda(0) = x \end{cases}$$

is given by $y_\lambda(t) = e^{-\lambda t}y$. Therefore, it is very easy to show that hypotheses 2.1, 2.2 and 2.3 are verified (the above parameterized class of control problems arises in the ergodic control of an affine stochastic partial differential equation see [7]).

For instance if $H = L^2([0,1])$, $\mathcal{D}(A) = H^2([0,1]) \cap H_0^1([0,1])$; $A = \frac{\partial^2}{\partial\zeta^2}$; $C = \sigma\frac{\partial}{\partial\zeta}$ with $\sigma^2 < 2$; $B = I$, then we obtain the following stochastic heat equation with diffused control:

$$\begin{cases} d_t y_\lambda(t,\zeta) = \left(\dfrac{\partial^2 y_\lambda}{\partial\zeta^2}(t,\zeta) - \lambda y_\lambda(t,\zeta)\right)dt + \\ \qquad\qquad + u(t,\zeta)dt + \dfrac{\partial y_\lambda}{\partial\zeta}(t,\zeta)d\beta_t \qquad \forall\zeta \in (0,1) \\ y_\lambda(t,0) = y(t,1) = 0 \\ y_\lambda(0,\zeta) = x(\zeta) \end{cases}$$

Moreover $(A, B, C)$ is $I$-stabilizable and, if $S = I$, $(A, \sqrt{S}, C)$ is detectable and all our assumptions are satisfied (see [6] and [4]).

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