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ABSTRACT

A nonlinear modal analysis is proposed to describe the dynamic behavior of nonlinear multi-degree-of-freedom systems. The solution is based upon the nonlinear mode superposition approach. The calculation of nonlinear natural frequencies and nonlinear normal modes of nonlinear structures obtained by using the single nonlinear mode approach is reviewed and compared to that obtained implicitly by the proposed approximation based upon the equivalent linearization approach. The aim of this work is to obtain a simple and rapid stationary solution which can be applied to real cases of large structures having nonlinear stiffness. In the experimental purpose, the new identification method of nonlinear modes obtained from forced responses is introduced. This approach is particularly important for large order systems for which a truncation of infinite modal coordinates to only a few lower modal coordinates can considerably reduce computational time. Some examples including experimental simulation have been introduced to illustrate the efficiency, the accuracy and the advantages of the proposed methods.

List of Symbols

- $[B]$ coordinate transformation matrix
- b_{jk} participation of nonlinear mode
- $[C]$ damping matrix
- $[C']$ equivalent damping matrix
- c' element of equivalent damping matrix
- $[D]$ stiffness matrix of the nonlinear relative system, $[D] = [\tilde{K}] + [K]$
- E error function
- F function defined in Eqs. (29-30)
- $\{f(\dot{u},u)\}$ nonlinear restoring forces vector
- i $(-1)^{1/2}$
- $[K]$ stiffness matrix of the system
- $[K']$ equivalent stiffness matrix
- $[\tilde{K}]$ stiffness matrix of the relative system
- $[\tilde{K}_{nl}]$ nonlinear stiffness matix of the relative system
- $[\tilde{K}']$ equivalent stiffness matix of the relative system
- k' element of equivalent stiffness matrix
- $\tilde{k}_{ij}$ element of matrix $[\tilde{K}]$
- $[M]$ mass matrix of the system
- $[\tilde{M}]$ mass matrix of the relative system
- $[\overline{M}]$ mass matrix of the nonlinear system
- $\tilde{m}_{ij}$ element of matrix $[\tilde{M}]$
- n number of degrees of freedom of the system
- $\{p(t)\}$ vector of nodal forces
- q_j modal amplitude
- $[S_r]$ residual flexibility matrix
- $\overline{S}_{im(\Omega,q)}$ nonlinear dynamic flexibility
- $S_{im(\Omega)}$ linear dynamic flexibility

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T time period

$\{u\}$ vector of nodal displacements

$\{y_j\}$ modal coordinate of mode j

Greek Symbols

ε difference between the nonlinear system and its equivalent linear system

λ eigenvalue of the system

$\{\phi_j\}$ linear normal mode

$\{\bar{\phi}_j\}$ nonlinear mode

Ω frequency of excitation

ω_j linear natural frequency

$\bar{\omega}_j$ nonlinear natural frequency

1. Introduction

The modal analysis of linear structures is well developed and is a powerful tool in dynamic analysis, but unfortunately most of the industrial structures exhibit nonlinearities in certain local regions or possibly in the whole of the structure. Many authors [1,2,3] show that the period of vibration decreased considerably with increasing amplitude of motion and many experimental results [2,3] show that the dynamic behavior of structures subjected to large amplitudes can be well described by Duffing's equation. The specific examples of localized nonlinearity can be found in structures with joints which may open or close during dynamic loading and in structures mounted on a yielding support, such as buildings which are not anchored to prevent the foundations being lifted during earthquakes. It is evident that assumption of linear behavior is totally inappropriate when analyzing these systems. The indiscriminate application of standard modal analysis to systems possessing false linearity can lead to erroneous analysis and the conclusion of dynamical behavior of the structure, even for weak nonlinearity. The solution of nonlinear models can be obtained relatively simply by numerical integration methods such as Runge-Kutta methods. However, to obtain a stationary solution of a large degree-of-freedom system, this procedure would be unreasonably time-consuming. The nonlinear systems of routine engineering practice, particularly those subjected to dynamic loading, have continued to be solved by linear methods. The nonlinear behavior generally is treated with only simple approximations. Because new materials are being invented and industrial structures, particularly aeronautical structures, developed, the assumption that the model is linear is inappropriate in the analysis of real structures. Nonlinear methods must be used in the design of real structures. The solutions of nonlinear systems for a single or a small degree-of-freedom have been studied intensively by many authors; examples are the perturbation method of Poincaré and Lindstedt, the asymptotic method of Krylov-Bogoliubov-Mitropolsky, the method of equivalent linearization and the Ritz-Galerkin method [4,5]. Some of these methods have been quite successfully applied to single or few degrees-of-freedom systems, but often fail and are quite difficult to apply in the analysis of dynamic systems with many degrees of freedom and strong nonlinearity.

The purpose of this study is to obtain a simple and rapid stationary solution, in regard to computational cost and the mathematical complexities, to the dynamic analysis of large structures having nonlinear stiffness. The solution is based upon the well known linear modal analysis. The nonlinear normal mode is used for transforming the set of n coupled equations from a physical base motion of an n degree-of-freedom system into a set of n uncoupled equations in a modal base system. The concept of nonlinear normal mode of a nonlinear mass spring system was studied initially by Rosenberg [6] and developed later by Szemplinska-Stupnicka [7,8]. It was shown that the mode of vibration in resonant conditions was close to the nonlinear normal mode and was not close to the linear normal mode. The authors [9-12] show that the nonlinear normal mode can be used to approach the solution to large systems having nonlinear stiffness. For the purposes of practical engineering this procedure can be used to reduce computational time considerably by using only a few of the lower modal coordinates.

In the dynamic analysis, the solution of the models is obtained by assuming that all the necessary parameters are known. In fact, in practical situations, most of the important parameters are not known, especially the nonlinear parameters, which are very difficult to determine. Most of the nonlinear identification procedures which exist at present have problems regarding their mathematical complexity, convergence rate, storage requirement and very long computation time. In the case of a single degree-of-freedom model, Parzygnat [3] shows that the nonlinear parameter of Duffing's equation can be identified with the aide of Krylov-Bogoliubov-Mitropolsky formulation. In the context of the extension of standard modal analysis, the nonlinear modal parameter identification procedure is introduced in the last part of this work. This procedure can be applied to large-degree-of-freedom systems.

In the first part of this paper, the calculation of nonlinear natural frequencies and nonlinear normal modes of nonlinear systems is studied and compared to those obtained implicitly by the equivalent linearization approach. The interpolation procedure is used to facilitate the utilization of nonlinear terms. The modal truncation technique is also proposed to reduce computation time. It will be shown in the last part of this study that the proposed nonlinear modal identification procedure can be applied without difficulty to predict and reconstruct the dynamic behavior of large nonlinear systems based on the forced responses.

2. General Equation and Solution in Linear Cases

A nonlinear n degree-of-freedom system subjected to a harmonic excitation of frequency Ω is considered. The second order differential equation corresponding to the motion of this system can be expressed by

$$[M]\{\ddot{u}\}+[K]\{u\}+\{f(\dot{u},u)\}=\{p(t)\} \tag{1}$$

$[M]$ is the mass matrix; $[K]$ is the linear stiffness matrix; $\{u\}$ is the vector of unknown spatial displacements; $\{f(\dot{u},u)\}$ is the vector of nonlinear restoring forces which depends on the spatial displacements; $\{p(t)\}$ is the vector of harmonic forces. The matrices $[M]$ and $[K]$ are real and positive definite.

It is well known that when Eq. (1) is linear, $\{f(\dot{u},u)\}=0$, the solution of this equation can be obtained by using the Rayleigh-Ritz procedure. This is accomplished by approximating $\{u\}$ by a linear combination of normal modes $\{\phi_j\}$ and modal coordinate $\{y_j(t)\}$

$$\{u(t)\}=\sum_{j=1}^{n}\{\phi_j\}\{y_j(t)\} \tag{2}$$

$\{\phi_j\}$ and $\{y_j\}$ are the normal mode and the modal coordinate of mode j respectively.

The normal modes, $\{\phi\}$, decouple the linear second order differential equation of motion Eq. (1) into a set of n uncoupled equations. This possibility is quite natural, due to the orthogonality properties of the eigenvalue proplem which is capable of transforming the set of n coupled equations in physical base motion of a n degree-of-freedom system into a set of n uncoupled equations in modal base system.

Using the base transformation of Eq. (2), the uncoupled equation of motion, Eq. (1), in the j^{th} mode becomes

$$\{m_j\}\{\ddot{y}_j(t)\}+\{k_j\}\{y_j(t)\}=\{p_j(t)\} \tag{3}$$

where

$$\{m_j\} = \{\phi_j\}^T [M]\{\phi_j\}$$
$$\{k_j\} = \{\phi_j\}^T [K]\{\phi_j\}$$
$$\{p_j(t)\} = \{\phi_j\}^T \{p(t)\}$$

By assuming that the solution is periodic with the same period as the period of excitation, Ω, the modal coordinate, $\{y_j(t)\}$, can be obtained by

$$-\Omega^2 m\{y_j(t)\} + k_j\{y_j(t)\} = \{p_j(t)\} \tag{4}$$

The unknown normal modes and natural frequencies associated with Eq. (1) can be obtained by the eigenvalue problem of

$$\lambda_j[M]\{\phi_j\} = [K]\{\phi_j\} \tag{5}$$

Were λ and ϕ are eigenvalues and eigenvectors. The natural frequencies of the system can be obtained by

$$\omega_j^2 = \lambda_j \tag{6}$$

The normal modes, ϕ, of the system correspond to the eigenvectors of Eq. (5).

3. Nonlinear Mode Superposition Method

In the case of a nonlinear system having nonlinear stiffness $\{f(\dot{u}, u)\} = \{f(u)\} \neq 0$, it is interesting to approximate the solution of nonlinear differential Eq. (1) using the notation analogous to that in the linear case. The nonlinear normal mode is used to transform the set of n coupled equations in a physical base into a set of n uncoupled equations in a modal base system [12]. This modal approximation is very important especially for systems with a large number of degrees-of-freedom, in which the truncation of the number of modal coordinates can be used to considerably reduce calculation time.

(a) Coordinate Transformation

The discretization of most physical systems described in Eq. (1) is often carried out using finite element methods. However, for the nonlinear system the nodal points are often interconnected by nonlinear elements whose behavior depends upon the relative coordinates between these points. Thus is it more convenient to transform mass matrix $[M]$ and stiffness matrix $[K]$ of Eq. (1) from absolute coordinates to matrices $[M]$ and $[K]$ in relative coordinates z so that they satisfy all of the geometric constraints on the system

$$z_{ij} = u_i - u_j \tag{7}$$

The vector of relative spatial displacements z can be achieved by a linear transformation

$$z = \left[B^{-1}\right]u \tag{8}$$

$[B]$ is the coordinate transformation matrix from an absolute coordinate system to a relative coordinate system.

The relative mass matrix, stiffness matrix and nonlinear restoring forces vector satisfy the conditions

$$\left[\tilde{M}\right] = [B]^T[M][B] \tag{9}$$

$$\left[\tilde{K}\right] = [B]^T[K][B] \tag{10}$$

$$\tilde{f}(z) = \sum \alpha z^r \tag{11}$$

r is the order of nonlinearity. Hence the general form of the transformation of the mass matrix and the stiffness matrix of Eqs. (9) and (10) will be

$$\tilde{m}(i,j) = \sum_{p=1}^{n}\sum_{q=1}^{n} m(p,q) \tag{12}$$

$$\tilde{k}(i,j) = \sum_{p=1}^{n}\sum_{q=1}^{n} k(p,q) \tag{13}$$

where $\tilde{m}(i,j)$, and $\tilde{k}(i,j)$ are the elements of matrices $[\tilde{M}]$ and $[\tilde{K}]$ respectively.

By the coordinate transformation of Eqs. (8) - (11), the second order differential equation of the autonomous conservative nonlinear system of Eq. (1) can be expressed by

$$\left[\tilde{M}\right]z + \left[\tilde{K}\right]z + \tilde{f}(z) = 0 \tag{14}$$

Analogous to that in the linear case, the solution of Eq. (14) can be approximated as a linear combination of n nonlinear normal modes $\tilde{\phi}_j(q_j)$ and n modal amplitudes q_j

$$\{z(t)\} = \sum_{j=1}^{n}\{\tilde{\phi}_j\}(q_j)\{\tilde{y}_j(t)\} \tag{15}$$

z is the vector of complex displacement amplitudes of the structure, $\{\tilde{\phi}_j\}(q_j)$ is the nonlinear normal mode of mode j which will be explained in the next section.

(b) Nonlinear Normal Mode

For the nonlinear lightly damped multi-degree-of-freedom system, the stationary solution in resonant conditions can be considered as a nonlinear normal mode. Thus the multi-degree-of-freedom system of Eq. (14) is reduced to the single-degree-of-freedom system described by the single resonant normal coordinates. Equation (15) in the single normal coordinate becomes

$$z(t) = \{\tilde{\phi}_j\}(q_j)\{\tilde{y}_j(t)\} \quad (16)$$

$\{\tilde{\phi}_j\}(q_j)$ and $\{\tilde{y}_j\}$ are, respectively, the normal mode and the modal coordinate of mode j in the relative base coordinate. When the force of excitation is periodic, the stationary response is generally periodic with the same period as the period of excitation Ω, thus at $\Omega = \Omega_j$, the single normal mode solution of Eq. (16) becomes

$$z(t) = \{\tilde{\phi}_j\}q_j \cos\Omega_j t \quad (17)$$

where q_j is the modal amplitude.

The assumption of a single mode in resonant conditions and a periodic solution in the nonlinear problem is used and approved for many classes of problems and in a wide range of nonlinearity even for systems that can be treated as strongly non-linear. The unknown nonlinear natural frequencies Ω_j and nonlinear normal modes $\tilde{\phi}_j$ can be obtained by inserting Eq. (17) into Eq. (14) and neglecting all the higher harmonic terms

$$[D](q_j)\{\tilde{\phi}_j\} = \bar{\lambda}_j[\tilde{M}]\{\tilde{\phi}_j\} \quad (18)$$

where

$$[D](q_j) = [\tilde{K}] + [\tilde{K}_{nl}](q_j) \quad (19)$$

and $[\tilde{K}_{nl}]$ (q_j) is the nonlinear stiffness matrix which depends on the modal amplitude q_j.

The eigenvalue problem expressed by Eq. (18) is not a standard linear form, thus in general, it can not be solved by the standard eigenvalue solution. The nonlinear eigenvalue problem of Eq. (18) can be treated by the following procedures which depend on the number of nonlinearities.

(1) Single Nonlinearity

The nonlinear eigenvalue problem of Eq. (18) can be linearized by setting a nonlinear element of the nonlinear eigenvalue problem of Eq. (14) to unity, when the system possesses only one nonlinearity. Thus the eigenvalues and eigenvectors of the nonlinear system can be obtained by the standard eigenvalue solution from the linearized eigenvalue problem of Eq. (18).

(2) Multi Nonlinearities

The nonlinear eigenvalue problem can be obtained only by numerical procedures, where the system possesses multi nonlinearities. There exist many numerical procedures to solve nonlinear problems. The Newton-Raphson procedure has been used for this study.

The nonlinear Eq. (18) can be rewritten in the following form

$$[\tilde{K}] + [\tilde{K}_{nl}](q_j) - \bar{\lambda}_j[\tilde{M}]\{\tilde{\phi}_j\} = g(\bar{\lambda}_j, \tilde{\phi}_j) \quad (20)$$

The n degree Eq. (20) possesses $n + 1$ unknowns of eigenvector and eigenvalue elements, thus a priori, Eq. (20) can not be used to obtain $n + 1$ unknowns. The unknowns should be arranged in order to render Eq. (20) resolvable. This can be done by setting one of the elements of the eigenvector to unity. Finally, the n unknowns of Eq. (20) can be obtained by the following procedure

$$s' = s - G\left(\bar{\lambda}_i, \tilde{\phi}_j\right)^{-1} g\left(\bar{\lambda}_j, \tilde{\phi}_j\right) \tag{21}$$

where

$$\begin{aligned} s^T &= \left(\bar{\lambda}_j \tilde{\phi}_{2j} \ldots \tilde{\phi}_{nj}\right) \\ G\left(\bar{\lambda}_j, \tilde{\phi}_j\right) &= dg\left(\bar{\lambda}_j, \tilde{\phi}_j\right) / ds \\ G\left(\bar{\lambda}_j, \tilde{\phi}_j\right)_{i1} &= dg\left(\bar{\lambda}_j, \tilde{\phi}_j\right)_i / d\bar{\lambda}_j = -\left[M_{i1}\right] \\ G\left(\bar{\lambda}_j, \tilde{\phi}_j\right)_{ip} &= dg\left(\bar{\lambda}_j, \tilde{\phi}_j\right)_i / d\tilde{\phi}_j = \left[K_{i,p}\right] + \bar{k}'\left(q_j\right) - \bar{\lambda}_j\left[M_{i,p}\right] \\ \bar{k}'\left(q_j\right) &= d\left[\bar{K}_{nl}\right]\left(q_j\right) / d\tilde{\phi}_j \end{aligned} \tag{22}$$

with $p = 2, 3, \ldots\ldots n$ and s is a vector of n unknowns of 1 eigenvalue and n -1 eigenvector elements.

Using the previous values of modal amplitude, this iterative procedure should converge rapidly. The linear natural frequencies and the linear normal modes can be used as the initial values. The nonlinear natural frequencies and the nonlinear normal modes of the nonlinear system are obtained as a function of modal amplitude q_j by progressively increasing the modal amplitude.

$$\bar{\omega}_j^2\left(q_j\right) = \bar{\lambda}_j \tag{23}$$

$$\left\{\bar{\phi}_j\right\}\left(q_j\right) = [B]\left\{\tilde{\phi}_j\right\} \tag{24}$$

where B is the coordinate transformation matrix, $\{\bar{\phi}_j\}$ (q_j) and $\{\tilde{\phi}_j\}$ are the nonlinear normal mode in the absolute coordinate system and in the relative coordinate system, respectively.

(c) Normalization of Nonlinear Modes

In order to simplify the utilization of nonlinear modal parameters, it is more convenient to normalize nonlinear modes $\bar{\phi}_j(q_j)$ as a linear combination of n associated linear normal modes $\bar{\phi}_{ij}$ and n participations of nonlinear modes $b_{jk}(q_j)$ in the following form

$$\left\{\bar{\phi}_{ij}\right\}\left(q_j\right) = \sum_{j=1}^{n} b_{jk}\left(q_j\right)\left\{\phi_{ik}\right\} \tag{25}$$

$$b_{jk}\left(q_j\right) = \left\{\phi_{ik}\right\}^T [M]\left\{\bar{\phi}_{ij}\right\}\left(q_j\right) \tag{26}$$

$$\forall q_j \;:\; b_{jk}\left(q_j\right) = 1 \tag{27}$$

The linear normal mode ϕ must satisfy the following orthogonality relationship

$$\begin{gathered} [\phi_i]^T [M] \{\phi_j\} = \delta_{ij} \\ i,\ j = 1,\ 2,\ \ldots,\ n \end{gathered} \tag{28}$$

where δ_{ij} is the Kronecker delta.

It is more convenient to utilize the nonlinear frequencies $\overline{\omega}_j(q_j)$ and the participations of nonlinear modes $b_{jk}(q_j)$ in continuous analytical form as a function of the modal amplitude q_j.

$$\overline{\omega}_j\left(q_j\right) = \overline{\omega}_j(0) + F_\omega\left(q_j\right) \tag{29}$$

$$b_{jk}\left(q_j\right) = F_b\left(q_j\right) \tag{30}$$

In the general case, $F_\omega(q_j)$ and $F_b(q_j)$ can be represented by the following real rational polynomial

$$F\left(q_j\right) = \frac{a_1 q_j^2 + a_2 q_j^4 + \ldots + a_M q_j^{2M}}{1 + b_1 a_j^2 + b_2 q_j^4 + \ldots + b_N q_j^{2N}} = \frac{P\left(q_j\right)}{Q\left(q_j\right)} \tag{31}$$

where $\overline{\omega}_j(0)$ is the nonlinear natural frequency at $q_j = 0$, which represent the associated linear natural frequency ω_j of the system and M, N are the degrees of numerator and denominator. The unknown coefficients $a_1, a_2, \ldots a_M$ and $b_1, b_2, \ldots b_N$ are determined in such a way that the error between the function $F(q_j)$ and the L measured values of $\overline{\omega}_j(q_j)$ or $b_{jk}(q_j)$ is minimized. Applying the least-squared error criterion to the function in Eq. (31) directly in order to find the unknown coefficients will result in highly nonlinear equations, which are extremely difficult to solve even using some iterative procedure. The weight function is used in order to improve the convergence of the method. The minimization between Eq. (31) and measured values is performed according to the criterion

$$E = \sum_{k=1}^{L} \left[w_k \left(F_k - D_k \right) \right]^2 = \text{minimum} \tag{32}$$

where w_k, F_k and D_k are the arbitrary weighting function, the function $F(q_j)$ and the measured value at modal amplitude q_j respectively.

Let $E(M, N)$ denote the sum of the weighted squared errors after t iterations and Q_k' refer to the value of Q_k at t - 1 iterations. With the weight set equal to Q_k/Q_k', $E(M, N)$ is defined as

$$E(M,N) = \sum_{k=1}^{L} \left(Q_k \,/\, Q_k' \left(F_k - D_k \right) \right)^2 \tag{33}$$

$$E(M,N) = \sum_{k=1}^{L} \left(P_k - D_k Q_k \right)^2 / \, Q_k'^2 \tag{34}$$

Here the squared error is weighted more evenly at all points to overcome the problem of poor fit. Setting the partial of $E(M, N)$ to zero with respect to a_j and b_j, one obtains a set of $M + N$ linear equations

$$\delta E(M,N)/\delta a_j = 0$$

$$\sum_{j=1}^{M}\left[\sum_{k=1}^{L} q_{jk}^{2(i+j)} / Q_k'^2\right]a_j - \sum_{j=1}^{N}\left[\sum_{k=1}^{L} D_k q_{jk}^{2(i+j)} / Q_k'^2\right]b_j = \sum_{k=1}^{L} D_k q_{jk}^{2i} / Q_k'^2 \tag{35}$$

$$i = 1,2,\ldots,M$$

$$\delta E(M,N)/\delta b_j = 0$$

$$\sum_{j=1}^{M}\left[\sum_{k=1}^{L} D_k q_{jk}^{2(i+j)} / Q_k'^2\right]a_j - \sum_{j=1}^{N}\left[\sum_{k=1}^{L} D_k^2 q_{jk}^{2(i+j)} / Q_k'^2\right]b_j = \sum_{k=1}^{L} D_k^2 q_{jk}^{2i} / Q_k'^2 \tag{36}$$

$$i = 1,2,\ldots,N$$

Equations (35) and (36) represent a set of linear equations of a_j and b_j which can be solved easily. The iterations are continued until the solution coverges.

(d) Nonlinear Mode Superposition Method

Analogous to the linear case, the solution of Eq. (1) can be approximated as a linear combination of n nonlinear normal modes $\bar{\phi}_j(q_j)$ and n modal amplitudes q_j.

$$\{u(t)\} = \sum_{j=1}^{n}\{\bar{\phi}_j\}(q_j)q_j(t)\cos\Omega t \tag{37}$$

where u_i is the complex displacement amplitude at point i of the structure, $\bar{\phi}_j(q_j)$ is the nonlinear normal mode of mode j obtained mode by mode by utilizing the single non-linear mode procedure described previously. Inserting Eq. (37) into the equation of motion, Eq. (1), and assuming that the solution is slightly coupled, the transformed equation of motion of Eq. (1) becomes

$$-\Omega^2\bar{m}_j q_j + i\Omega\bar{c}_j q_j + \bar{k}_j q_j + \bar{f}_j(q_j) = \bar{p}_j \tag{38}$$

where

$$\bar{m}_j = \{\bar{\phi}_j\}(q_j)^T[M]\{\bar{\phi}(q_j)\}; \qquad c_j = \{\bar{\phi}_j(q_j)\}^T[C]\{\bar{\phi}(q_j)\}; \qquad k_j = \{\bar{\phi}_j(q_j)\}^T[K]\{\bar{\phi}(q_j)\}$$
$$\bar{f}_j(q_j) = \{\bar{\phi}_j(q_j)\}^T f_j(q_j) = \{\bar{\phi}(q_j)\}^T[K_{nl}]\{\bar{\phi}(q_j)\}; \qquad \bar{p}_j = \{\bar{\phi}_j(q_j)\}^T[P] \tag{39}$$

and $[K_{nl}]$ is the nonlinear stiffness part of the system.

The complex modal amplitude q_j of each mode j of Eq. (38) can be expressed by

$$q_j = \frac{\{\bar{\phi}_j(q_j)\}^T [P]}{\bar{m}_j\left(\bar{\omega}_j^{\,2}(q_j) - \Omega^2\right) + i\Omega\bar{c}_j} \tag{40}$$

where the linear and the nonlinear stiffness of the system in modal base are approximated by

$$\bar{k}_j q_j + \bar{f}(q_j) = \bar{m}_j \bar{\omega}_j^2(q_j) \tag{41}$$

Inserting Eqs. (25) and (26) into Eq. (39) and regarding the orthogonality relationship of Eq. (28), the complex modal amplitude q_j of Eq. (40) becomes

$$q_j = \frac{\sum_{k=1}^{n} b_{jk}(q_j)\phi_{\mathrm{mk}} P_m}{\left(\bar{\omega}_j^2(q_j) - \Omega^2\right)\sum_{k=1}^{n} b_{jk}(q_j)^2 + i\Omega \sum_{k=1}^{n} b_{jk}(q_j)^2 c_j} \tag{42}$$

where

$$\bar{m}_j = \sum_{k=1}^{n} b_{jk}(q_j)^2; \qquad \bar{c}_j = \sum_{k=1}^{n} b_{jk}(q_j)^2 c_j$$
$$c_j = \{\phi_j\}^T [C] \{\phi_j\} \tag{43}$$

and $\bar{m}_j$, and $\bar{c}_j$, and $\bar{\phi}_j$ are the nonlinear modal mass, the nonlinear modal damping and the linear normal mode respectively. The solution of Eq. (42) can only be found numerically, and a set of n modal amplitudes are obtained for each step of the iteration. In general, the required solution will be found very rapidly.

(e) Modal Truncation

The modal superposition procedure can be used effectively for large degrees-of-freedom and only a few of the lower modal coordinates need be employed. However, this truncation can induce unacceptable errors in the solution of dynamic problems. Therefore the linear residual flexibility can be introduced in order to reduce the errors which appear due to modal truncation.

The dynamic flexibility expression of the n degree-of-freedom subjected to a harmonic excitation Ω can be expressed by

$$u_i = \bar{S}_{im}(\Omega) F_m \tag{44}$$

The dynamic flexibility $\bar{S}_{im}(\Omega)$ is a function of the frequency of excitation Ω, and expressed by

$$\bar{S}_{im}(\Omega) = \sum_{j=1}^{n} \frac{\bar{S}_j(\Omega)}{\left(1 - \Omega_2/\bar{\omega}_j^2\right) - i\Omega\bar{c}_j/\bar{m}_j\bar{\omega}_j^2} \tag{45}$$

with

$$\bar{S}_j(\Omega) = \frac{\phi_j^T \phi_j}{\bar{m}_j \bar{\omega}_j^2} \tag{46}$$

where $\bar{S}_j(\Omega)$ is the effective flexibility of mode j. The dynamic flexibility matrix of the structure tends toward static flexibility S_0 when the frequency of excitation tends to zero. If only p modes are taken into account in the solution of dynamic problems, the residual flexibility matrix of a truncated model can be approximated by static effect in the form

$$\bar{S}_r = S_0 - \sum_{j=1}^{p} \bar{S}_j = \sum_{j=p+1}^{n} \bar{S}_j \tag{47}$$

If n is taken to be sufficiently large, the residual flexibility matrix S_r becomes small and the errors in the solution can be made toward zero. By using p first of the lower modal corrdinates and residual flexibility matrix of Eq. (47), the truncated model of Eq. (45) becomes

$$\bar{S}_{im}(\Omega) = \sum_{j=1}^{p} \frac{\bar{S}_j(\Omega)}{\left(1 - \Omega^2/\bar{\omega}_j^2\right) - i\Omega\bar{c}_j/\bar{m}_j\bar{\omega}_j^2} + [s_r] \tag{48}$$

where $[S_r]$ is the linear residual flexibility matrix of Eq. (47).

4. Comparative Method

The equivalent linearization method and the incremental harmonic balance method which take into account the participation of all resonant modes have been chosen for comparison. It will also be shown that nonlinear resonant frequencies $\bar{\omega}_j(q_j)$ and nonlinear modes $\bar{\phi}_j(q_j)$ of the structure can be obtained implicitly from forced responses by the equivalent linearization method and compared to those obtained by the single nonlinear mode approach.

(a) Equivalent Linearization Method

The principal approach of the method is replacement of the nonlinear differential equation by a linear equation for which the exact analytical formula for the solution is known. Consider the nonlinear second order differential equation of motion of the nonconservative nonlinear n degree-of-freedom with nonlinear vector function $f(\dot{u},u)$. The equivalent linear system of system Eq. (1) may be expressed in the form [13]

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} + [C']\{\dot{u}\} + [K']\{u\} = g(t) \tag{49}$$

where $[C']$ and $[K']$ are respectively the equivalent damping matrix and the equivalent stiffness matrix in the relative base coordinates.

By assuming a periodic solution, the general solution of Eq. (49) can be written in Fourier series in the form

$$u_i(t) = \sum_{j=1}^{n} a_{ij} \cos j\omega t + b_{ij} \sin j\omega t \tag{50}$$

which in this approximation is truncated to a single harmonic function

$$u_i(t) = a_i \cos \omega t + b_i \sin \omega t \tag{51}$$

The matrices $[C']$ and $[K']$ are determined by minimizing the difference ε between the nonlinear system and the equivalent linear system Eq. (49) for every $u(t)$ where ε is the equation difference

$$\varepsilon = \{f(\dot{u},u)\} - [C']\{\dot{u}\} - [K']\{u\} \tag{52}$$

The minimization of ε is performed according to the criterion

$$E(\varepsilon'\varepsilon) = \frac{1}{T}\int_0^T \varepsilon^2 dt = \text{minimum} \qquad T = 2\pi/\omega \tag{53}$$

The necessary conditions for the minimization specified by Eq. (53) are

$$\frac{\delta}{\delta c_{ij}'} E(\varepsilon'\varepsilon) = 0 \tag{54}$$

$$\frac{\delta}{\delta k_{ij}'} E(\varepsilon'\varepsilon) = 0 \tag{55}$$

where c_{ij}' and k_{ij}' are the elements of the matrices C' and K' respectively.

For the system with local nonlinearities, the equivalent linear terms c' and k' may be constructed according to Eqs. (53), (54) and (55) as

$$c_{ij}' = \frac{\int_0^T \dot{u} f(\dot{u},u)dt}{\int_0^T \dot{u}^2 dt} \tag{56}$$

$$k_{ij}' = \frac{\int_0^T \{u\}\{f(\dot{u},u)\}dt}{\int_0^T \{u^2\}dt} \tag{57}$$

where c_{ij}' and k_{ij}' are obtained iteratively for each step of calculation of the equivalent linear system. In the case of nonlinear damping equal to zero and nonlinear stiffness on mass n, the equivalent stiffness of Eq. (57) is simplified to [12]

$$k_n' = 3/4 a_n \left(a_n^2 + b_n^2\right) \tag{58}$$

The nonlinear resonant frequencies $\overline{\omega}(q)$ and the nonlinear mode $\overline{\phi}(q)$ of the forced nonlinear system can be calculated by a standard eigenvalue solution for each equivalent stiffness value. The n modal-amplitudes correspond to the nonlinear terms $\overline{\omega}(q)$ and $\overline{\phi}(q)$ can be obtained by

$$\{q\} = \left[\overline{\phi}\right]^{-1} \{u\} \tag{59}$$

Thus, for each step of calculation, a set of n nonlinear natural frequencies $\overline{\omega}(q)$ and n nonlinear normal modes $\overline{\phi}(q)$ are obtained as a function of their modal-amplitude q.

(b) Incremental Harmonic Balance Method

The method is based on the harmonic balance method. The incremental procedure is used to improve the convergence of the method [14, 15]. The set of linearized equations can be obtained and solved at each incremental step. Consider the nonlinear second order differential equation of motion of the nonlinear n degree-of-freedom. The solution is assumed to be analogous to that of the equivalent linearization method Eq. (50).

Assume the U_0 denotes the current state of vibration and ΔU the corresponding increment. Then a neighboring state can be written as

$$u = u_0 + \Delta u \tag{60}$$

with

$$\Delta u_i(t) = \sum_{j=1}^{n} \Delta a_{ij} \cos j\omega t + \Delta b_{ij} \sin j\omega t \tag{61}$$

Inserting Eq. (60) into Eq. (1) and neglecting all the terms containing increment products, a linearized incremental equation is obtained in the form

$$\begin{aligned} &-\Omega^2[M]\{\Delta U\} + i\Omega[C]\{\Delta U\} + [K]\{\Delta U\} + f\left(\Delta\dot{U}, \Delta U\right) = \{R\} \\ &\{R\} = \Omega^2[M]\{U\} - i\Omega[C]\{U\} - [K]\{U\} - f\left(\dot{U}, U\right) \end{aligned} \tag{62}$$

R is a corrective term and will vanish when the required solution is reached.

Equation 62 is linear but it has variable coefficients and thus is difficult to solve. In accordance with the Galerkin procedure, Eq. (62) can be replaced by

$$\int_0^T \left[-\Omega^2[M]\{\Delta U\}+\Omega[C]\{\Delta U\}+[K]\{\Delta U\}+\left\{F\left(\Delta\dot{U},\Delta U\right)\right\}\right]\begin{Bmatrix}\cos\omega t\\ \cdots\cdots\\ \sin\omega t\end{Bmatrix}d(\Omega t)=$$

$$\int_0^T R\begin{Bmatrix}\cos\omega t\\ \cdots\cdots\\ \sin\omega t\end{Bmatrix}d(\Omega t) \tag{63}$$

An incremental system of $2n$ linear equations in terms of the Δa_{ij} and Δb_{ij} is obtained from Eq. (63), hence, at every iteration, the increments Δa_{ij} and Δb_{ij} can be obtained by

$$\{\Delta a,\Delta b\}=[K]^{-1}\{R\} \tag{64}$$

where $\{\Delta a,\Delta b\}$, $[K]$ and $\{R\}$ are respectively the vector of all the increment terms, the tangent matrix and the corrective vector. The neighboring state of Eq. (60) is then obtained by adding these corresponding increments to their current state. By repeating the incremental procedure, the required solution can be obtained finally.

5. Numerical Example

A discrete model of n degrees with cubic nonlinearity has been chosen as an example to illustrate the advantages and the accuracy of the methods. A mass spring system of Fig. 1 with n masses, n viscous dampings, n linear springs, a cubic nonlinear spring and a single harmonic excitation of frequency Ω at mass n is considered. The linear stiffness matrix $\tilde{K}$ and nonlinear stiffness matrix $\tilde{K}_{nl}$ in the relative coordinates are obtained by the following form

$$\tilde{K}=\begin{bmatrix}k_1 & & & \\ & k_2 & & \\ & & \cdot & \\ & & & k_n\end{bmatrix} \qquad \tilde{K}_{nl}=\begin{bmatrix}\alpha_1 & & & \\ & \alpha_2 & & \\ & & \cdot & \\ & & & \alpha_n\end{bmatrix}$$

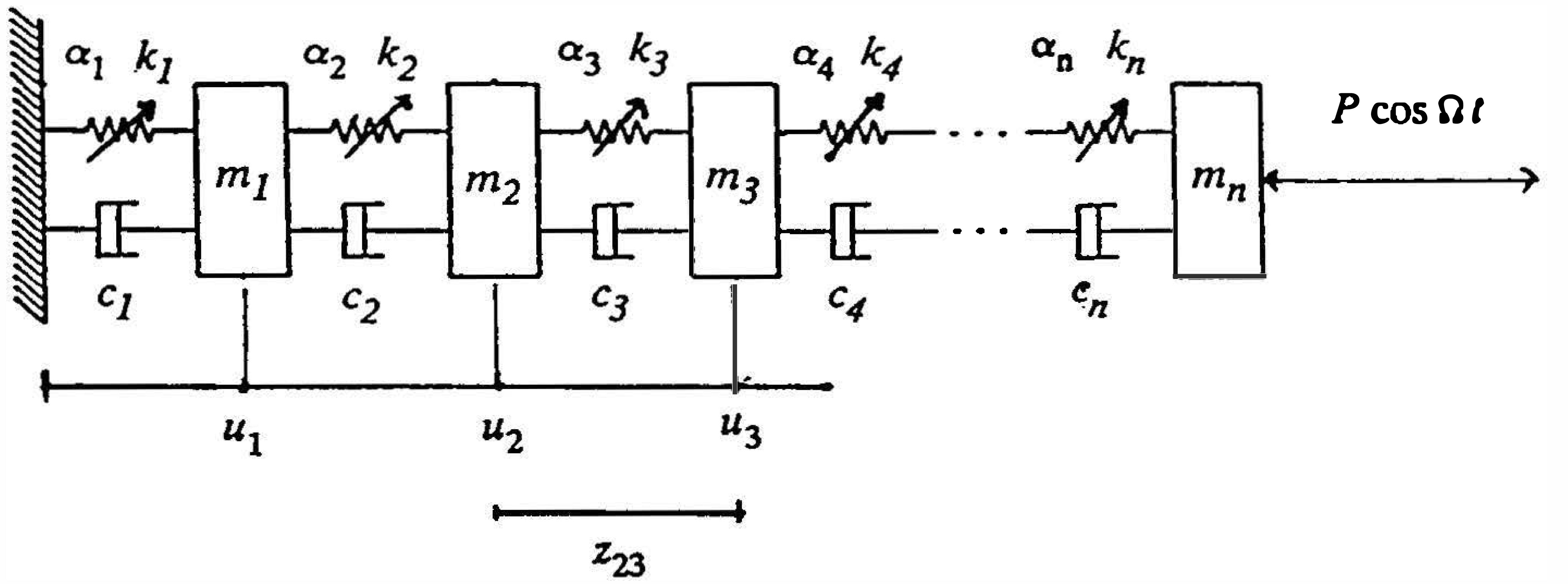


Fig.1 Model of n degrees of freedom system, n = 5; α_1 = 0.01 N/m^3; α_2-α_5 = 0.00 N/m^3; m_1-m_5 = 1.00 kg; c_1-c_5 = 0.02 $N\ sec/m$; k_1-k_5 = 1.00 N/m

The model parameters, shown in Fig. 2 and Fig. 3, are calculated by the nonlinear single mode method and the equivalent linearization method. A good approximation is shown. Figure 4 shows the frequency responses of mass 1 obtained by the Nonlinear Mode Superposition method and compared to that obtained by the Equivalent Linearization method and Incremental Harmonic Balance method. It has been shown that the nonlinear mode superposition method gives very satisfactory results. The frequency responses for various levels of force amplitude calculated by the Nonlinear Mode Superposition method are shown in Fig. 5.

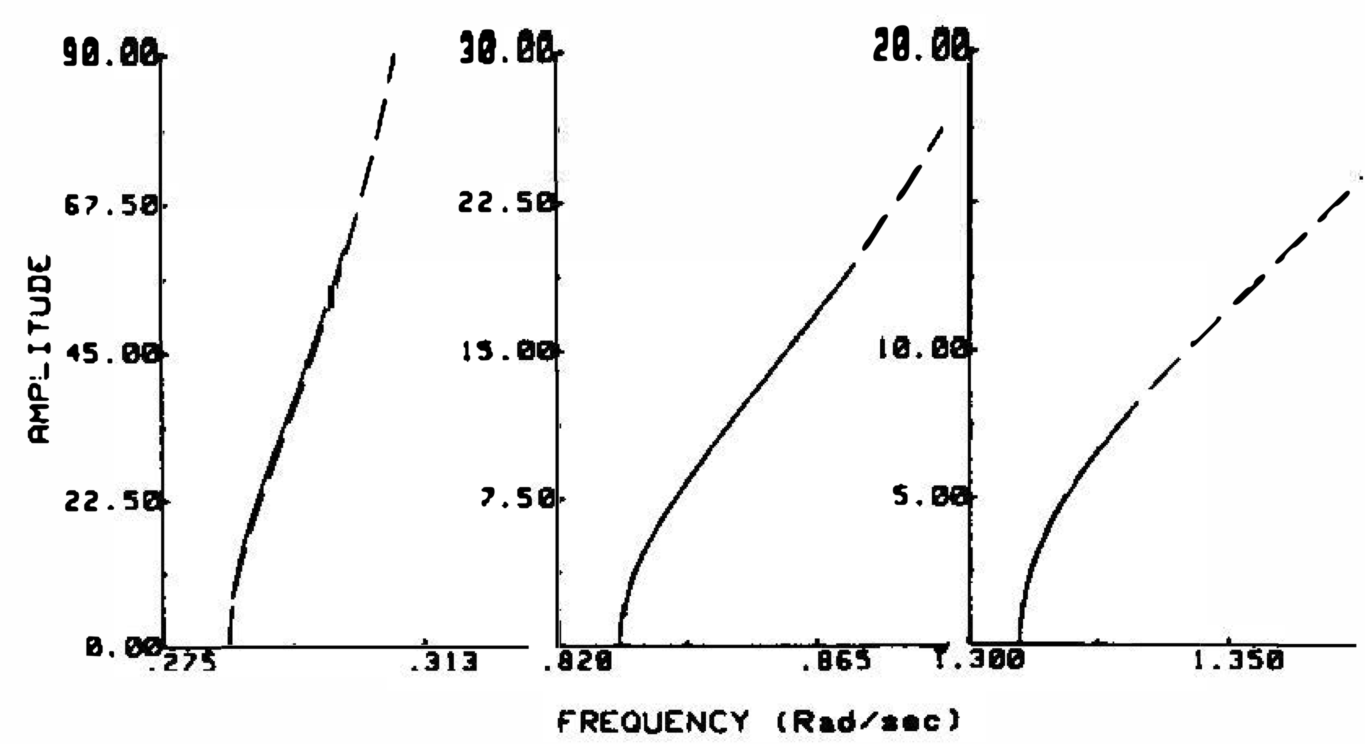


Fig. 2 The nonlinear natural frequency $\overline{\omega}_i(q_i)$ of modes 1, 2 and 3 as a function of modal amplitude q. (--- Nonlinear Single mode method, —— Equivalent Linearization method.)

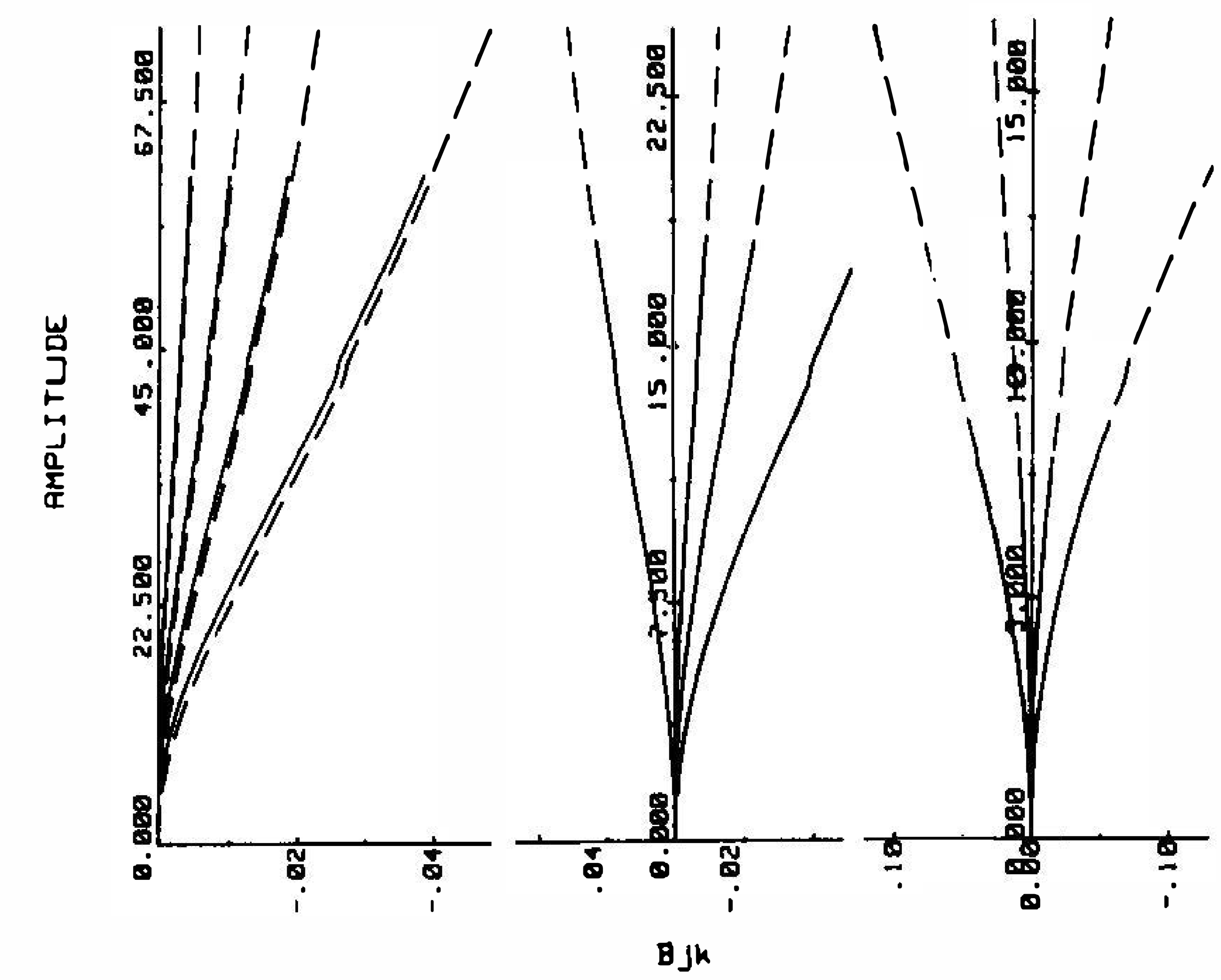


Fig. 3 The participations of nonlinear modes $b_{jk}(q_i)$ of mode 1, 2 and 3 as a function of modal amplitude q. (--- Nonlinear Single Mode method, —— Equivalent Linearization method.)

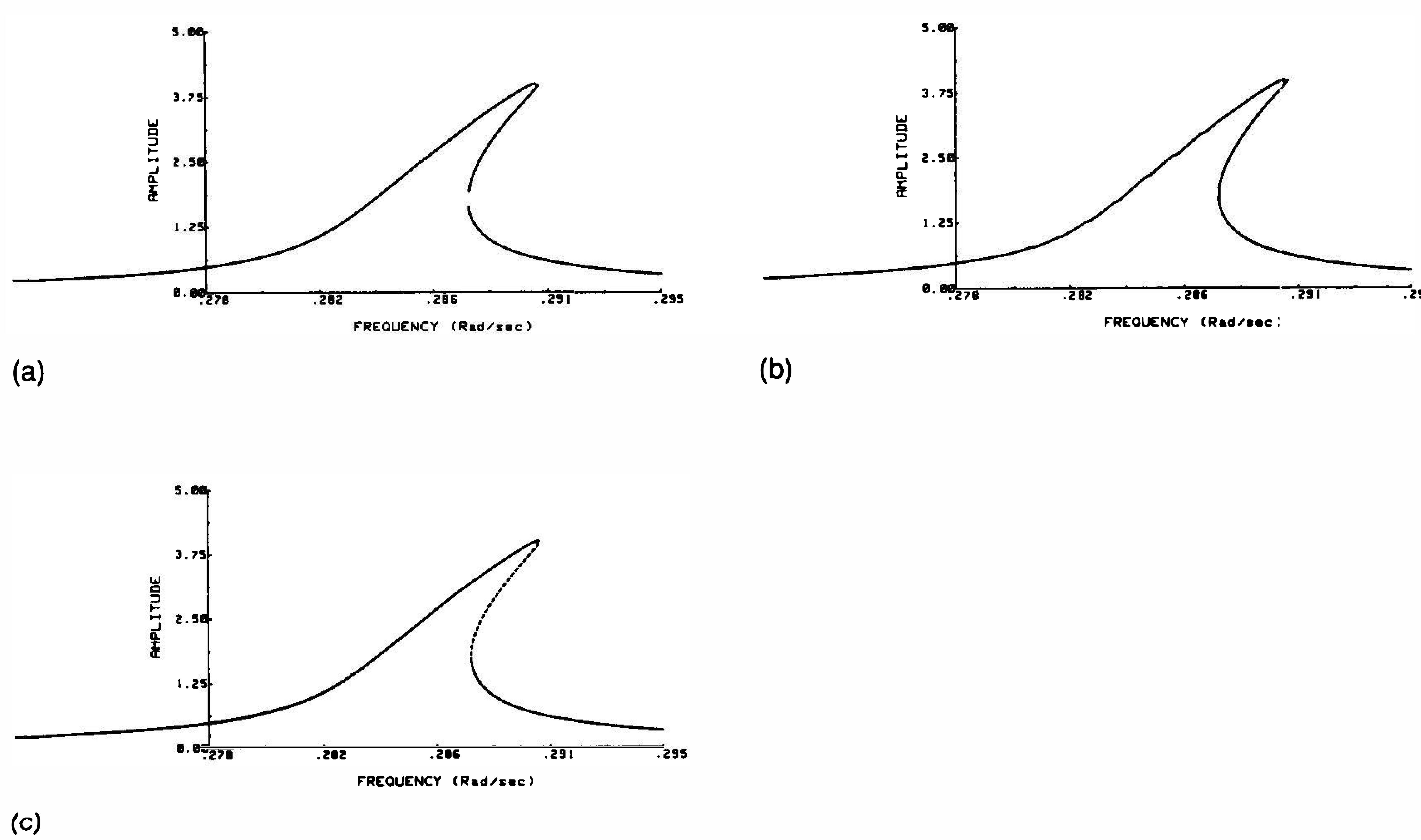


Fig. 4 Frequency response of mass 1 of mode 1 with amplitude force P = 0.02 N applied at mass 5. a) Nonlinear Mode Superposition method, b) Equivalent Linearization method, c) Incremental Harmonic Balance method.

The solutions to the dynamic analysis of a nonlinear system are obtained by assuming that all the necessary parameters are known in the previous parts. In practical situations most of the important parameters are not known, and these parameters have to be determined experimentally using the forced responses. The identified parameters can be utilized to predict and reconstruct the dynamic behavior of the nonlinear systems. Therefore the nonlinear modal parameter identification procedure is introduced for this purpose.

In accordance with the modal synthesis method, the dynamic behavior of the stationary solution of the nonlinear multi-degree-of-freedom system subjected to a harmonic excitation at point m can be obtained by the following dynamic flexibility expression

$$u_i = \bar{S}_{im}(\Omega, q) P_m \tag{65}$$

where u_i is the complex displacement amplitude at every point i of the structure. The nonlinear dynamic flexibility $\bar{S}_{im}(\Omega,q)$ is a function of the frequency Ω and the amplitude of the excitation, expressed by

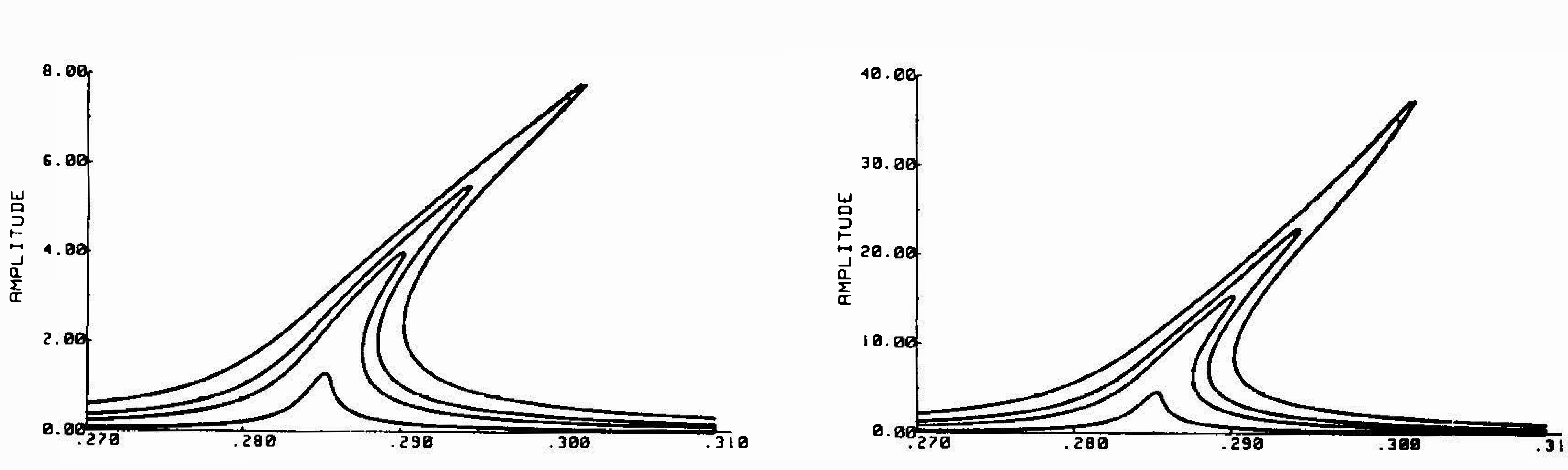
$$\bar{S}_{im}(\Omega, q) = \sum_{j=1}^{n} \frac{\bar{\phi}_{ij}(q_j)\bar{\phi}_{mj}(q_j)}{\bar{m}\left(\bar{\omega}_j^2(q_j) - \Omega^2\right) + i\Omega\bar{c}_j} \tag{66}$$

By inserting nonlinear normal mode $\bar{\phi}_{ij}(q_j)$, modal mass $\bar{m}_j$ and modal damping $\bar{c}_j$ of Eqs. (25) and (43) into Eq. (66), a nonlinear dynamic flexibility of Eq. (66) becomes

$$\bar{S}_{im}(\Omega, q) = \sum_{j=1}^{n} \frac{\sum_{l=1}^{n}\sum_{k=1}^{n} b_{jl}(q_j) b_{jk}(q_j) \phi_{il}\phi_{mk}}{\left(\bar{\omega}_j^2(q_j) - \Omega^2\right)\sum_{l=1}^{n} b_{jl}(q_j)^2 + i\Omega \sum_{l=1}^{n} b_{jl}(q_j)^2 c_j} \tag{67}$$

(a) Nonlinear Single-Degree-of-Freedom (NLSDOF)

The nonlinear modal parameters can be obtained satisfactorily from forced responses neighboring only one resonant condition when the resonant frequencies are not close to each other. The participation of the



(a) (b)

Fig. 5 Frequency response of mode 1 for various levels of amplitude of excitation. *P* = 0.006 *N*, 0.02 *N*, 0.03 *N*, and 0.05 *N* respectively. ______ Nonlinear Mode Superposition method, ... Equivalent Linearization method. a) Frequency response of mass 1, b) Frequency response of mass 5

non-resonant mode is relatively small, thus it can be replaced by the linear modal parameters.

The stationary solution of Eq. (65) can be approximated, when the frequency of excitation Ω is very close to the resonant frequency $\overline{\omega}_j(q_j)$ in the following form

$$u_i = \left[\overline{S}_{im}(\Omega, q) + S_{im}(\Omega)\right] P_m \tag{68}$$

The dynamic flexibility $S_{im}(\Omega)$ represents a linear part of the non-resonant mode

$$S_{im}(\Omega) = \frac{\phi_{il}\phi_{ml}}{\left(\omega_l^2 - \Omega^2\right) + i\Omega c_l} \qquad l \neq j \tag{69}$$

The linear resonant frequencies ω_l and linear normal mode ϕ_{ij} are calculated by standard modal analysis in the case of low amplitude. The participation of the resonant mode is given by

$$\overline{S}_{im}(\Omega, q) = \frac{\sum_{l=1}^{n}\sum_{k=1}^{n} b_{jl}(q_j) b_{jk}(q_j) \phi_{il}\phi_{mk}}{\left(\overline{\omega}_j^2(q_j) - \Omega^2\right)\sum_{l=1}^{n} b_{jl}(q_j)^2 + i\Omega \sum_{l=1}^{n} b_{jl}(q_j)^2 c_j} \tag{70}$$

The unknowns modal parameters $b_{jk}(q_j)$ and $\overline{\omega}_j(q_j)$ are obtained without difficulty by the numerical method for each frequency of excitation. The linear resonant frequencies ω_j and linear normal mode ϕ_{ij} are calculated by standard modal analysis [10,11] in the case of low amplitude.

(b) Nonlinear Multi-Degree-of-Freedom (NLMDOF)

The participation of the neighboring resonance is not negligible and their modal amplitude no longer small where the resonant frequencies are close to each other. Assuming that the participation is linear can lead to an erroneous result. Thus, the participation of the neighbor resonant must be considered nonlinear.

The nonlinear modal parameters can be extracted by the iteration procedures. Thus, by repeating such a process at each mode, it is possible to obtain the set of parameters which constitutes Eq. (67). The linear modal parameters can be introduced as the initial values. For each iteration, the nonlinear resonant frequencies $\overline{\omega}_j(q_j)$ and the participations of nonlinear modes $b_{jk}(q_j)$ must be performed in continuous form with the aid of Eqs. (29) and (30). This algorithm should be a rapidly iterative procedure based upon the previous values for each frequency of excitation.

(c) Example

A continuous system having the nonlinear contraints shown in Fig. 6 has been chosen as an experimental simulation. It consists of a uniform bar of length l, cross sectional area S, mass m and Young's modulus E, which is clamped at one end and anchored at the other by a nonlinear spring. The static displacement-force curve of the nonlinear spring is shown in Fig. 7. The identified results shown in Fig. 8 are calculated from the forced responses of the continuous system with two levels of amplitude of excitation. The frequency response shown in Fig. 9 is obtained with the aid of the identified nonlinear mode with a harmonic force imposed at $l = 0.20$ m. A good approximation has been shown.

7. Conclusion

The nonlinear mode superposition approach has been presented in describing the dynamic behavior of nonlinear multi-degree-of-freedom systems. As has been shown, the principal advantages of the methods are that it is simple and easy to apply to large structures thanks to the matrix form of the proposed method. The calculation time can be reduced considerably by truncating the modal coordinates to only a few lower modal coordinates. The accuracy of the methods appears to be well within the purposes of practical engineering. The capability of the nonlinear modal identification method to predict and to reconstruct the dynamic behavior of the nonlinear system without knowing the numbers, the values and the regions of nonlinearities is one of the great advantages of the methods.

Because of the availability of powerful computers and the low price of computational time in recent years, the utilization of the proposed methods in describing the dynamic behavior of the stationary solution of large nonlinear structures appears very promising because of its computational efficiency and especially mathematical simplicity.

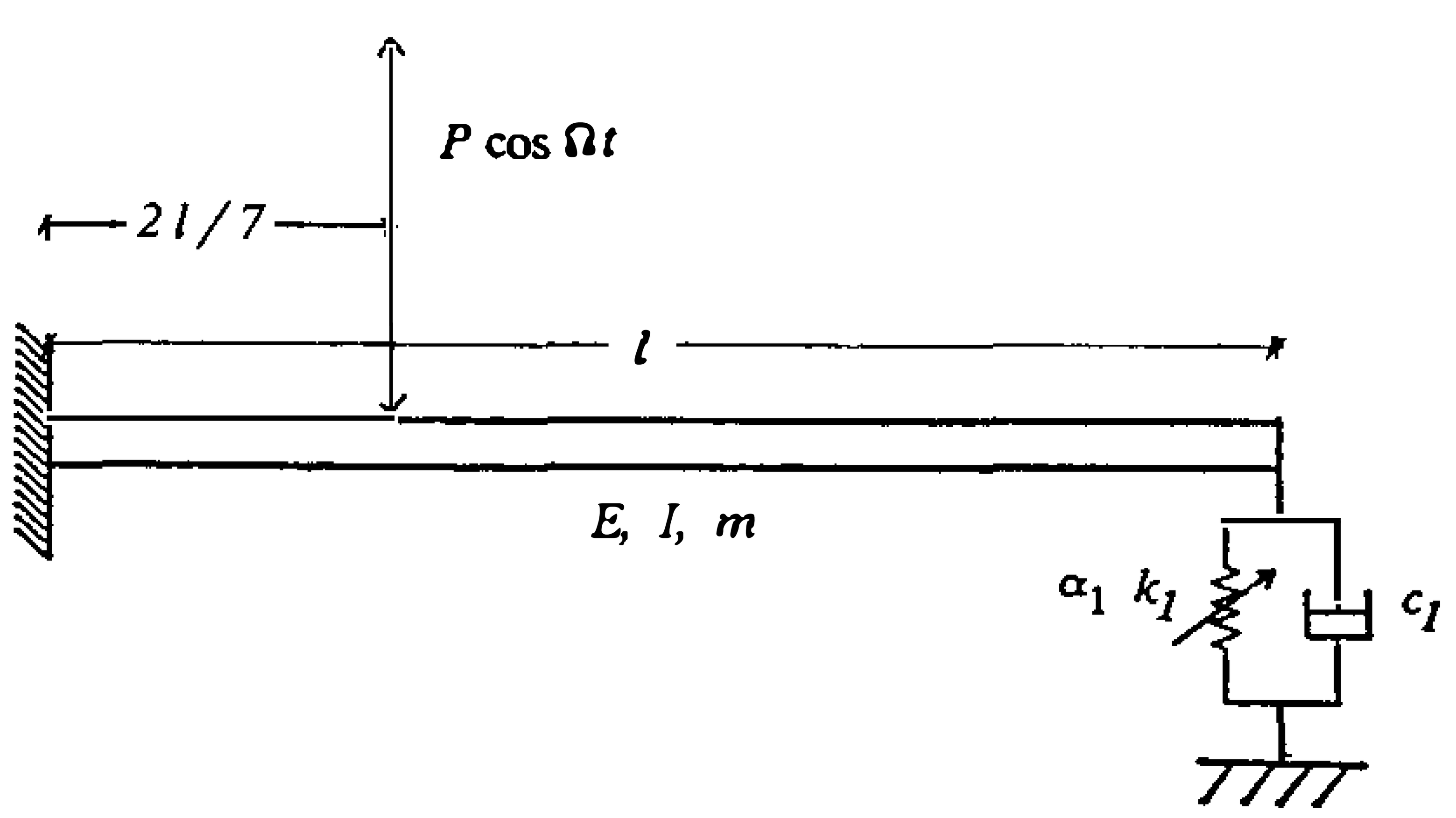


Fig. 6 Model of a continous system. l **= 0.70** m**;** m **= 3.120** kg/m**;** I **= 1.3333** $10^{-8} m^4$**;** c_1 **= 1.00** N/m^3**;** E **= 2.1** 10^{11} N/m^2**;** k_l **= 1973.03** N/m**;** α_l **= 38.23** 10^7 N/m^3

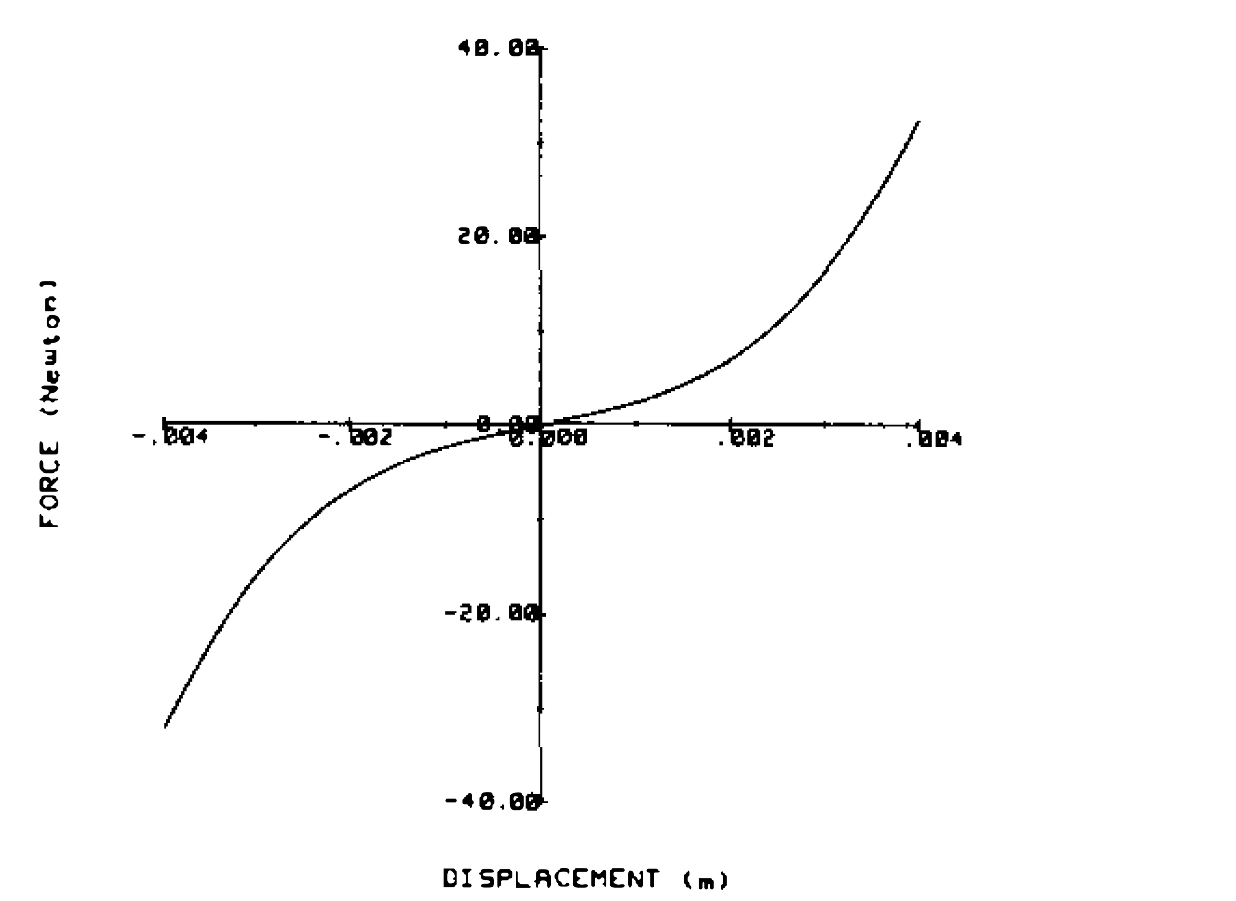


Fig. 7 Static displacement force of nonlinear spring

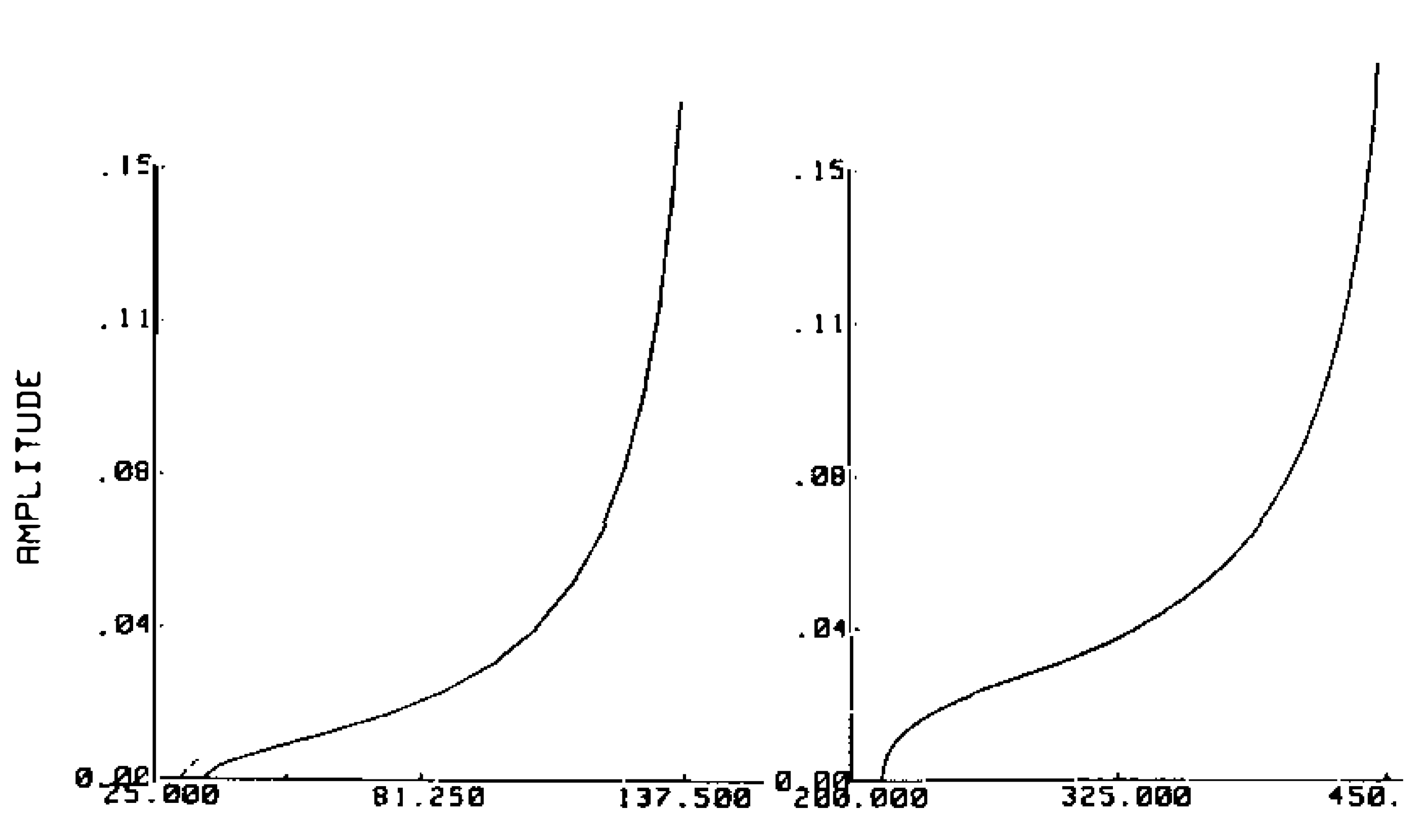


Fig. 8 The nonlinear natural frequency $\overline{\omega}_j(q_j)$ **as a function of modal amplitude** q_j

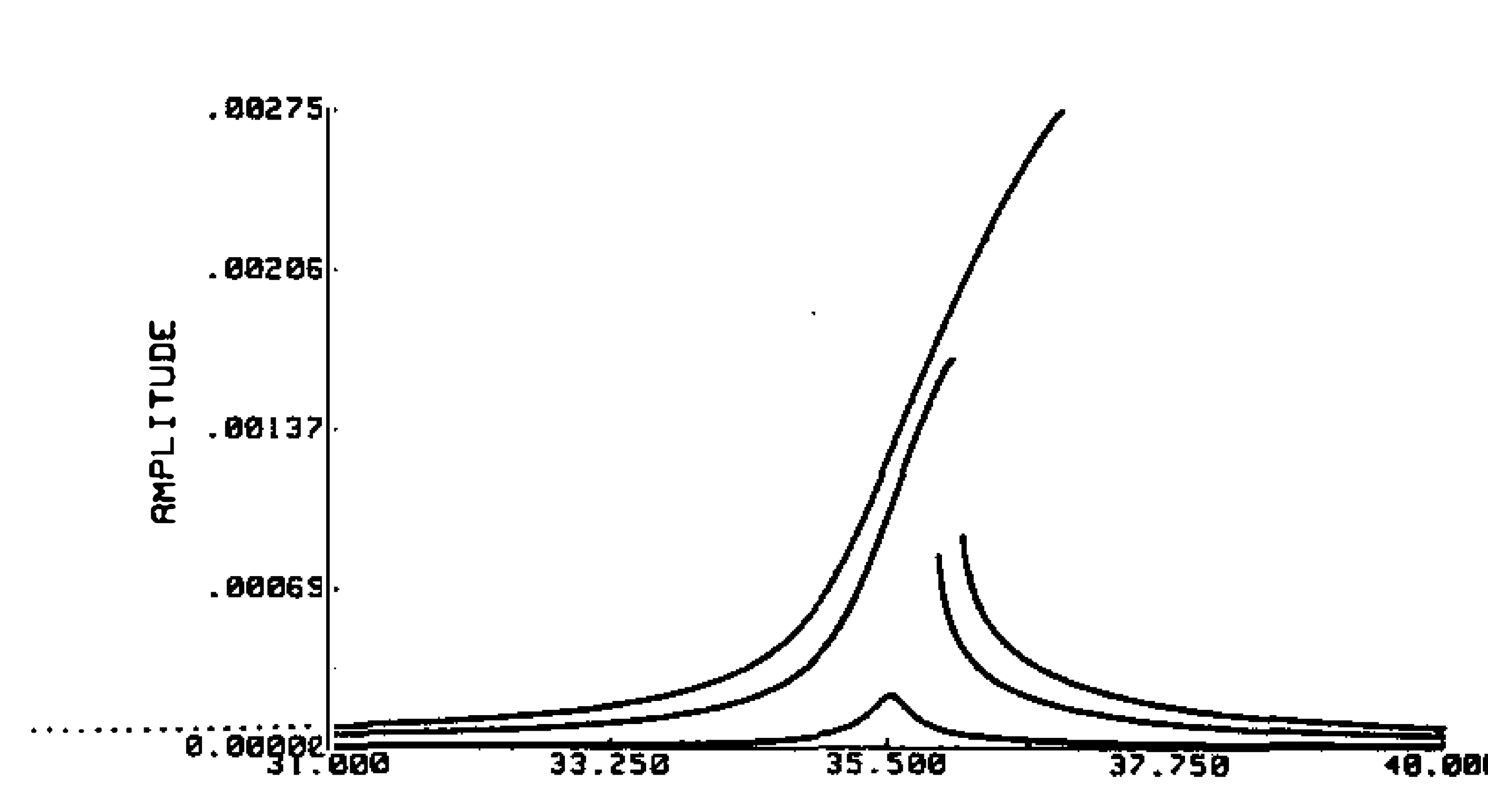


Fig. 9 Frequency response of the continuous system of mode 1. ...... Modal Sup. ____ Identified response

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The following awards were presented at IMAC 10, February 3-6, 1992, San Diego, CA.

1992 DOMINICK J. DEMICHELE Award

This award, established in 1989 to honor Dominick J. DeMichele, recognizes an individual who, in the opinion of the SEM Honors Committee, has demonstrated one or more of the following attributes: exemplary service and support to the modal analysis community demonstrated through participation in educational seminars, as a speaker at the International Modal Analysis Conferences (IMAC), or through administrative service on an SEM-IMAC steering committee which contributes to the on-going educational enrichment of SEM and the modal analysis community. The person need not be a member of SEM.

Presented to

Prof. David L. Brown, University of Cincinnati

International Journal of Analytical and Experimental Modal Analysis Best Paper Award

Nominees for this annual award are selected by a papers review committee composed of the Journal's Technical and Associate Editors. The Committee submits a ranked list of papers from the preceding volume year to the SEM Honors Committee. Final selection is made by the Honors Committee at their Fall Meeting.

Presented to

Prof. J. D. Jones, Purdue University

and

Prof. C. R. Fuller, Virginia Polytechnic Institute and State University

for

Active Control of Structurally Coupled Sound Fields in Elastic Cylinders by Vibrational Force Inputs

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