TUTORIAL: SIGNAL PROCESSING ASPECTS OF STRUCTURAL IMPACT TESTING

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ABSTRACT

Spectra estimated from structural impulse tests via fast Fourier transform (FFT) algorithms are susceptible to data acquisition related difficulties. The problem stems from vibration response signals which decay sufficiently slow that it is impractical to capture the waveform in its entirety. The problem is further compounded by the data acquisition and processing constraints imposed by FFT equipment. This paper reviews and investigates various aspects of FFT based signal processing in relation to commonly encountered situations in structural impact testing. The analysis is based on time and spectral relationships developed for a single degree-of-freedom (SDOF) system excited by a half sine force impulse. The truncation problem is investigated and the results show that provided at least six time constants of data are captured the error in the estimated frequency response function magnitude and phase is less than 0.5%. Next, the coloring effects of exponential windowing on spectra from an idealized impact test are examined. The results indicate that the windowing effects can be usually compensated for and accurate modal parameters extracted. Finally, the "double hit" phenomena is examined to illustrate the effects on the spectra and the implications with respect to modal analysis procedures.

List of Symbols

c	viscous damping constant (N-sec/m)
$E[\ ]$	expectation operator
F_0	amplitude of half sine forcing function for single hit (N)
F_1	amplitude of first half sine forcing function during double hit event (N)
F_2	amplitude of second half sine forcing function during double hit event (N)
$F(t)$	forcing function
$F(\omega)$	Fourier transform of forcing function
$F(\omega,T_1)$	finite Fourier transform of forcing function
f_n	natural frequency (Hz)

$f_{analysis}$	analysis frequency (Hz)
f_{sample}	digital sample rate (Hz)
Δf	frequency resolution (Hz)
$G_{FF}^T(\omega,T_1)$	auto spectrum of transient force
$G_{xx}^T(\omega,T)$	auto spectrum of vibration response signal
$G_{Fx}^T(\omega,T)$	cross spectrum of force and vibration response signal
$G_{yy}^T(\omega,T)$	auto spectrum of general transient signal
$H(\omega)$	theoretical frequency response function

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$H(\omega,T)$ estimated frequency response function

    i $\sqrt{-1}$

   k spring constant (N/m)

  m mass (kg)

   n number of spectral lines

   R modal residue

$sinc(f)$ $sine(f)/f$

   T capture time (sec)

  T_1 half sine input force pulse duration (sec)

    t time (sec)

$u(t-\gamma_0)$ unit step function, active when argument is ≥ 0

W_{BOX} boxcar data capture window

W_{EXP} exponential data capture window

$X(\omega)$ Fourier transform of system response

$X(\omega,T)$ finite Fourier transform of system response

   x system response (m)

$Y(\omega,T)$ finite Fourier transform of a general time signal

$y(t)$ general time signal

Greek

α integration variable

γ_0 time delay between impulses (sec)

ζ damping ratio

ζ_{cor} corrected damping ratio

τ_w exponential window time constant (sec)

τ_s single degree of freedom system time constant

ω frequency (rad/s)

ω_d damped natural frequency (rad/s)

ω_f half sine pulse forcing frequency (π/T_1) (rad/s)

ω_n undamped natural frequency (rad/s)

$*$ complex conjugate operator

1 Introduction

Impact excitation is one of the most common methods used for experimental modal testing. Hammer impacts produce a broadbanded excitation signal ideal for modal testing with a minimal amount of equipment and set up. Furthermore, it is versatile, mobile and produces reliable results. Although it has limitations with respect to precise positioning and force level control, overall its advantages greatly outweigh its disadvantages making it extremely attractive and effective for many modal testing situations [1].

The use of impulse testing with FFT signal processing methods presents data acquisition conditions which must be considered to ensure that accurate spectral functions are estimated. Problems stem from the availability of only a finite duration sample of the input and output signals. When a structure is lightly damped the response to the hammer impact may be sufficiently long that it is impractical to capture the entire signal. The truncation effect manifests itself in terms of a spectral bias error having the potential to adversely affect the estimated spectra.

The signal truncation problem is further compounded in practice by the computational and hardware constraints of the FFT processing equipment. Typically the equipment has a limited number of data capture lengths or frequency ranges which are available for an operator to select. Normally a user is more concerned with useable analysis frequencies and less with the data capture length. Therefore, it is conceivable that an inappropriate data capture duration could be used which truncates the vibration signal and introduces errors in the estimated spectra. To suppress the truncation a common practice is to artificially force it to decay within the data capture window [1,2,3]. This artificial reduction is obtained by multiplying the slowly decaying vibration signal by an exponential function. However, the application of the exponential window must be considered carefully since it may also adversely affect the estimated spectra.

A phenomena commonly encountered during impact testing is the so called "double hit". The "double hit" applies two impulses to the structure, one initially and one time delayed. Both the temporal and spectral characteristics of the "double hit" input and output are significantly different than a "single hit". The input force spectrum for the "double hit" no longer has the wide band constant type characteristics of a single hit. If not handled properly, the estimated spectral quantities may be significantly affected creating testing problems.

The purpose of this paper is to examine the use of impact vibration testing in relation to the constraints imposed by typical FFT signal processing techniques. The characteristics of the impact testing procedure are examined with analytical time and spectral functions developed for an idealized test: a single degree-of-freedom system excited by a half sine impact force. Once an understanding of the fundamental characteristics is developed it is applied to examine the specific situations encountered in structural impact testing. The relationship of the system's parameters with respect to data capture requirements is evaluated. The effects of exponential windowing is developed to examine the effects on the estimated spectra and modal parameters. Finally, the "double hit" phenomena is examined by combining the results from the single degree-of-freedom system excited by two impulses, one of which is time delayed. The results from these related studies are combined to provide insight into data acquisition guidelines for structural impact testing.

2. Single Degree-of-Freedom System Response to a Hammer Impulse

The response of a single degree of freedom viscously damped system to an impact can be determined by modeling the hammer impulse as a half sine pulse [4]:

$$
f(t) = \begin{cases} F_0 \sin\left(\dfrac{\pi}{T_1} t\right) & 0 \le t \le T_1 \\ 0 & T_1 < t \end{cases}
\tag{1}
$$

The vibration response of the SDOF system to this force is:

$0 \le t \le T_1$

$$
x(t) = e^{-\zeta \omega_n t} \left\{ \frac{F_0 \omega_f \left[\omega_f^2 m + c\zeta\omega_n - k \right]}{\omega_d \left[\left(k - \omega_f^2 m\right)^2 + \left(c\omega_f\right)^2 \right]} \sin(\omega_d t) \right.
$$

$$
\left. + \frac{c\omega_f F_0}{\left(k - \omega_f^2 m\right)^2 + \left(c\omega_f\right)^2} \cos(\omega_d t) \right\}
\tag{2}
$$

$$
+ \frac{F_0}{\left(k - \omega_f^2 m\right)^2 + \left(c\omega_f\right)^2} \left[\left(k - \omega_f^2 m\right) \sin(\omega_f t) - c\omega_f \cos(\omega_f t) \right]
$$

$$T_1 < t \leq T\infty$$

$$x(t) = e^{-\zeta \omega_n t} \left\{ \frac{F_0 \omega_f \left[\omega_f^2 m + c \zeta \omega_n - k \right]}{\omega_d \left[\left(k - \omega_f^2 m \right)^2 + \left(c \omega_f \right)^2 \right]} \sin(\omega_d t) \right.$$

$$\left. + \frac{c \omega_f F_0}{\left(k - \omega_f^2 m \right)^2 + \left(c \omega_f \right)^2} \cos(\omega_d t) \right\}$$

$$+ e^{-\zeta \omega_n (t - T_1)} \left\{ \frac{F_0 \omega_f \left[\omega_f^2 m + c \zeta \omega_n - k \right]}{\omega_d \left[\left(k - \omega_f^2 m \right)^2 + \left(c \omega_f \right)^2 \right]} \sin\left[\omega_d (t - T_1) \right] \right.$$

$$\left. + \frac{c \omega_f F_0}{\left(k - \omega_f^2 m \right)^2 + \left(c \omega_f \right)^2} \cos\left[\omega_d (t - T_1) \right] \right\}$$

A typical hammer impact and system response using these relationships are shown in Fig. 1. The similarity of the half sine pulse to actual impact hammer force signals is apparent along with the slow transient vibration response.

The Fourier transforms of the Eqs. (1) and (2) provide a means to develop frequency domain spectral functions.

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt$$

(3)

Note the integration can be performed only when analytical expressions exist for $x(t)$ and $f(t)$. In practice, these integrals cannot be evaluated because only finite duration digital representations of the waveforms exist. The duration, T, over which the samples are acquired can have a very profound effect on the Fourier transforms. Typically the choice of a capture duration, T, is not a parameter directly considered. For example, the signals would be digitized over a particular capture duration at a sample rate selected by the operator. Usually, the operator selects the desired frequency range which automatically defines the sample rate and capture duration. For a baseband analysis, the relationship between the analysis frequency, bandwidth (Δf), number of spectral lines (n) and the data capture duration (T) is:
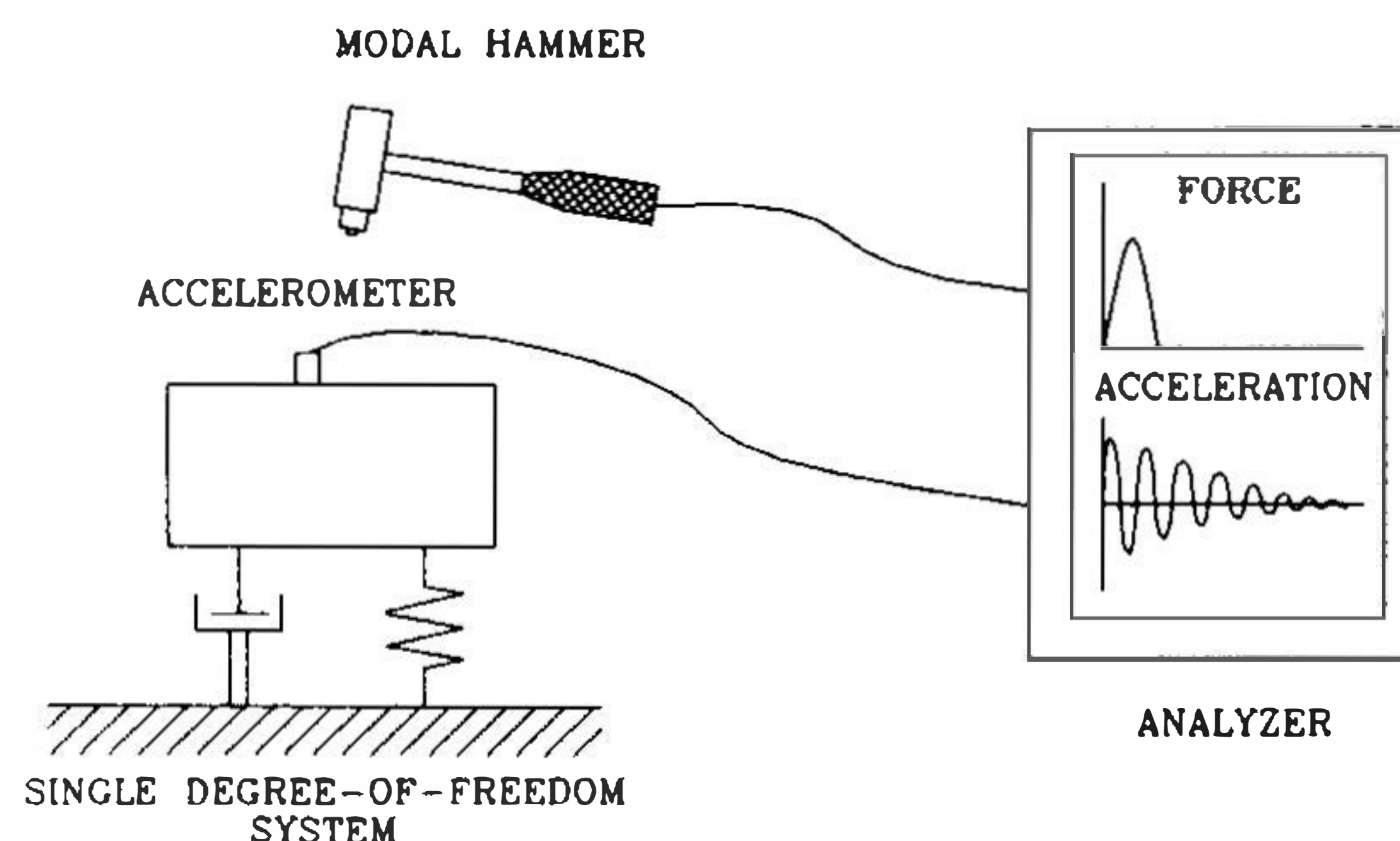
Fig. 1   Idealized impact test of a single degree-of-freedom system.

$$T = \frac{1}{f_{analysis}/n} = \frac{1}{\Delta f} \tag{4}$$

Equation (4) shows that the higher the frequency range (0 - $f_{analysis}$ Hz), the shorter the data capture duration, T.

The effects due to truncation can be evaluated by imposing a boxcar data capture window to the force and response signals. A boxcar data capture window of duration T is expressed as

$$W_{BOX}(t) = \begin{cases} 1.0 & 0 \le t \le T \\ 0 & T < t \end{cases} \tag{5}$$

The Fourier transform for a finite duration sample of a time signal, $y(t)$, is expressed as

$$Y(\omega, T) = \int_{-\infty}^{\infty} W_{BOX}(t) y(t) e^{-i\omega t} dt \tag{6}$$

Equation (6) may be simplified by changing the limits of integration

$$Y(\omega, T) = \int_{0}^{T} y(t) e^{-i\omega t} dt \tag{7}$$

The effects of signal truncation on the frequency domain functions for the idealized impact test can be evaluated by applying Eq. (7) to Eqs. (1) and (2). Assuming that the duration of the impact is small ($T_1 < T$) the force signal is entirely captured. Applying Eq. (7) to Eq. (1) the finite Fourier transform of the impact force is

$$F(\omega, T_1) = \frac{F_0 \omega_f}{\omega_f^2 - \omega^2} \left[1 + e^{-i\omega T_1} \right] \tag{8}$$

Equation (8) shows that the force energy spectrum is not a function of T and is unaffected by the window duration provided the entire impact signal is captured.

Similarily, the time response, Eq. (2), may be evaluated by Eq. (7)

$$\begin{aligned}
X(\omega, T) = & \frac{A_1 e^{i\alpha}}{\left[-\zeta\omega_n - i(\omega_d + \omega) \right]} \left\{ -1 + e^{\left[-\zeta\omega_n - i(\omega_d + \omega) \right]T} \right. \\
& \left. + e^{\left[-\zeta\omega_n - i(\omega_d + \omega) \right]T + (\zeta\omega_n + i\omega_d)T_1} - e^{-i\omega T_1} \right\} \\
& + \frac{A_1 e^{-i\alpha}}{\left[-\zeta\omega_n + i(\omega_d - \omega) \right]} \left\{ -1 + e^{\left[-\zeta\omega_n + i(\omega_d - \omega) \right]T} \right. \\
& \left. + e^{\left[-\zeta\omega_n + i(\omega_d - \omega) \right]T + (\zeta\omega_n - i\omega_d)T_1} - e^{-i\omega T_1} \right\} \\
& + \frac{A_2 e^{-i\beta}}{i(\omega_f + \omega)} \left[e^{-i(\omega_f + \omega)T_1} - 1 \right] \\
& - \frac{A_2 e^{i\beta}}{i(\omega_f - \omega)} \left[e^{i(\omega_f - \omega)T_1} - 1 \right]
\end{aligned} \tag{9}$$

where

$$A_1 = \frac{1}{2} \sqrt{\frac{\left[F_0 \omega_f \left(\omega_f^2 m + c\zeta\omega_n - k\right)\right]^2 + \left[\omega_d c\omega_f F_0\right]^2}{\left\{\omega_d \left[\left(k - \omega_f^2 m\right)^2 + \left(c\omega_f\right)^2\right]\right\}^2}}$$

$$\alpha = \tan^{-1}\left[\frac{\omega_f^2 m + c\zeta\omega_n - k}{\omega_d c}\right]$$

$$A_2 = \frac{1}{2} \frac{F_0}{\sqrt{\left(k - \omega_f^2 m\right)^2 + \left(c\omega_f\right)^2}}$$

$$\beta = \tan^{-1}\left[\frac{k - \omega_f^2 m}{c\omega_f}\right]$$

The finite Fourier transform of the vibration response is a function of the dynamic system parameters (m,c,k), the force characteristics (F_0, T_1) and the capture duration (T).

The Fourier transforms in Eqs. (8) and (9) can now be used to estimate spectra and the frequency response function. Since the force and vibration response signals are transient in nature the auto spectra may be estimated in terms of energy [5]

$$G_{yy}^T(\omega,T) = 2E\left[Y^*(\omega,T)Y(\omega,T)\right] \tag{10}$$

For deterministic signals Eq. (10) collapses to

$$G_{yy}^T = 2Y^*(\omega,T)Y(\omega,T) \tag{11}$$

Applying Eq. (11) to Eq. (8) the auto spectrum of the half sine pulse signal is

$$G_{FF}^T(\omega,T_1) = \frac{4F_0^2\omega_f^2}{\left(\omega_f^2 - \omega^2\right)}\left[1 + \cos(\omega T_1)\right] \tag{12}$$

The auto spectrum of the vibration response signal is

$$G_{xx}^T(\omega,T) = 2X^*(\omega,T)X(\omega,T) \tag{13}$$

The operations in Eq. (13) do not produce a compact form and will be evaluated numerically in subsequent sections.

The Frequency Response Function (FRF) can be formed to describe the causal relationship between the force (input) and vibration response (output). In practice, the frequency response function is estimated by the ratio of the cross spectrum between the two signals and the input auto spectrum. Since the signals in this study are noise free and deterministic the function may be determined directly from the Fourier transforms in Eqs. (8) and (9)

$$H(\omega.T) = \frac{G_{Fx}(\omega,T)}{G_{FF}(\omega,T)} = \frac{X(\omega,T)}{F(\omega,T)} \qquad (14)$$

Due to the resulting complexity of Eq. (14) it will be evaluated numerically.

To provide a comparison basis to evaluate truncation effects, it is necessary to express the unbiased spectral functions. Provided the force signal is not truncated, Eq. (12) represents an unbiased spectrum. The unbiased vibration spectrum can be evaluated by combining Eqs. (9) and (13) with the variable T (the data capture duration) set to infinity. The actual frequency response function can be developed directly from the single degree-of-freedom system parameters

$$H(\omega) = \left\{ \frac{\dfrac{1}{2m\omega_d}}{(\omega_d - \omega) + i\zeta\omega_n} - \frac{\dfrac{1}{2m\omega_d}}{(-\omega_d - \omega) + i\zeta\omega_n} \right\}, \qquad (15)$$

The functions developed in this section provide the foundation to evaluate the effects of truncation in an idealized structural impact test.

3. Truncation Effects on Frequency Response Functions

The expressions developed in Section 2 can be used to examine effects on spectra estimated from impact tests where the vibration decays sufficiently slowly that it can not be captured in its entirety. Consider a single degree-of-freedom system, Fig. 1, with a low natural frequency and damping ratio (f_n=10 Hz, ζ=0.01). The time constant for this second order system is given by
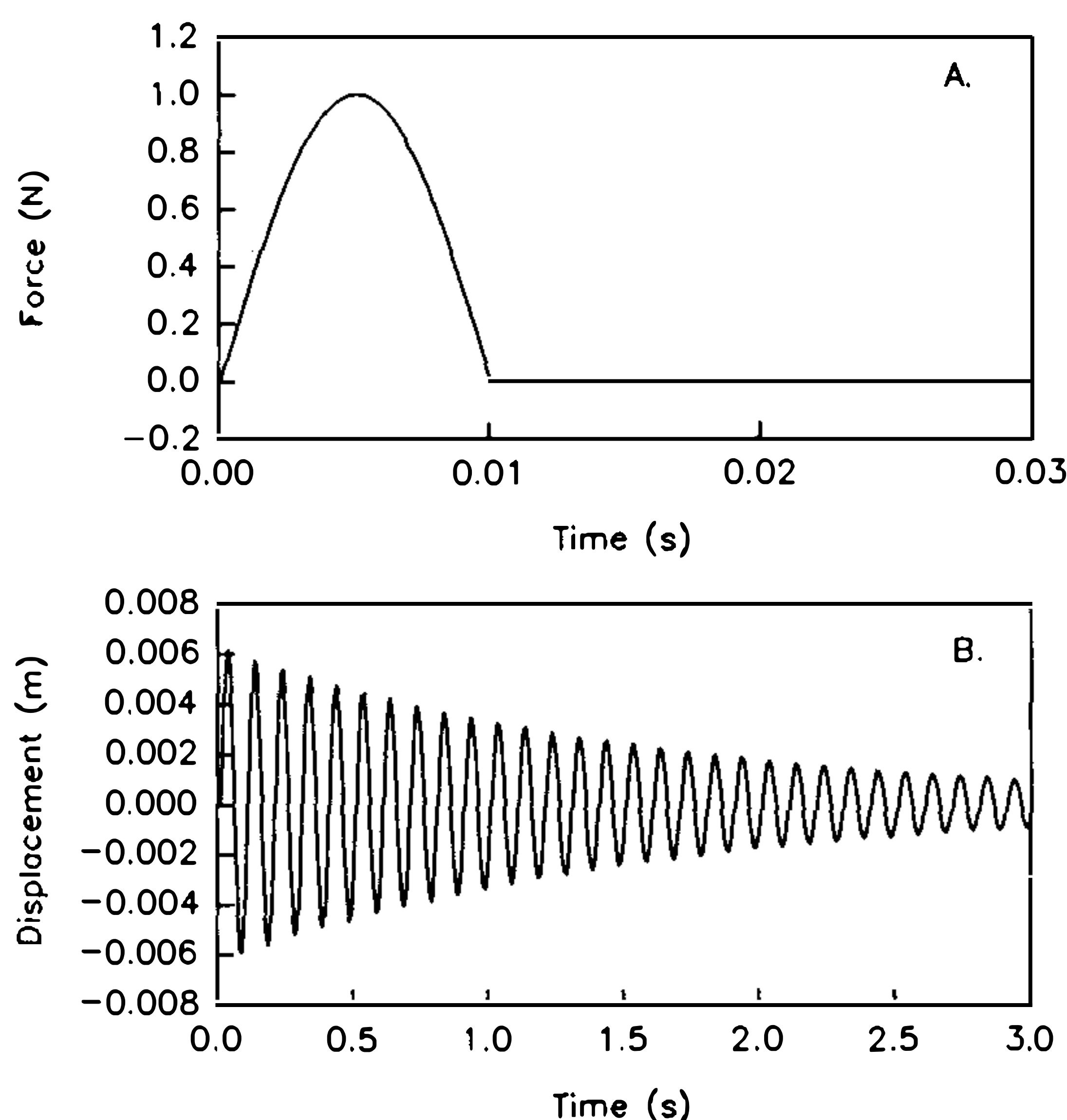
Fig. 2 a) Half sine impact force, T_1=0.01 (sec) b) Vibration response of a single degree-of-freedom (f_n=10 Hz, ζ=0.01) to the impact force in Fig. 2a
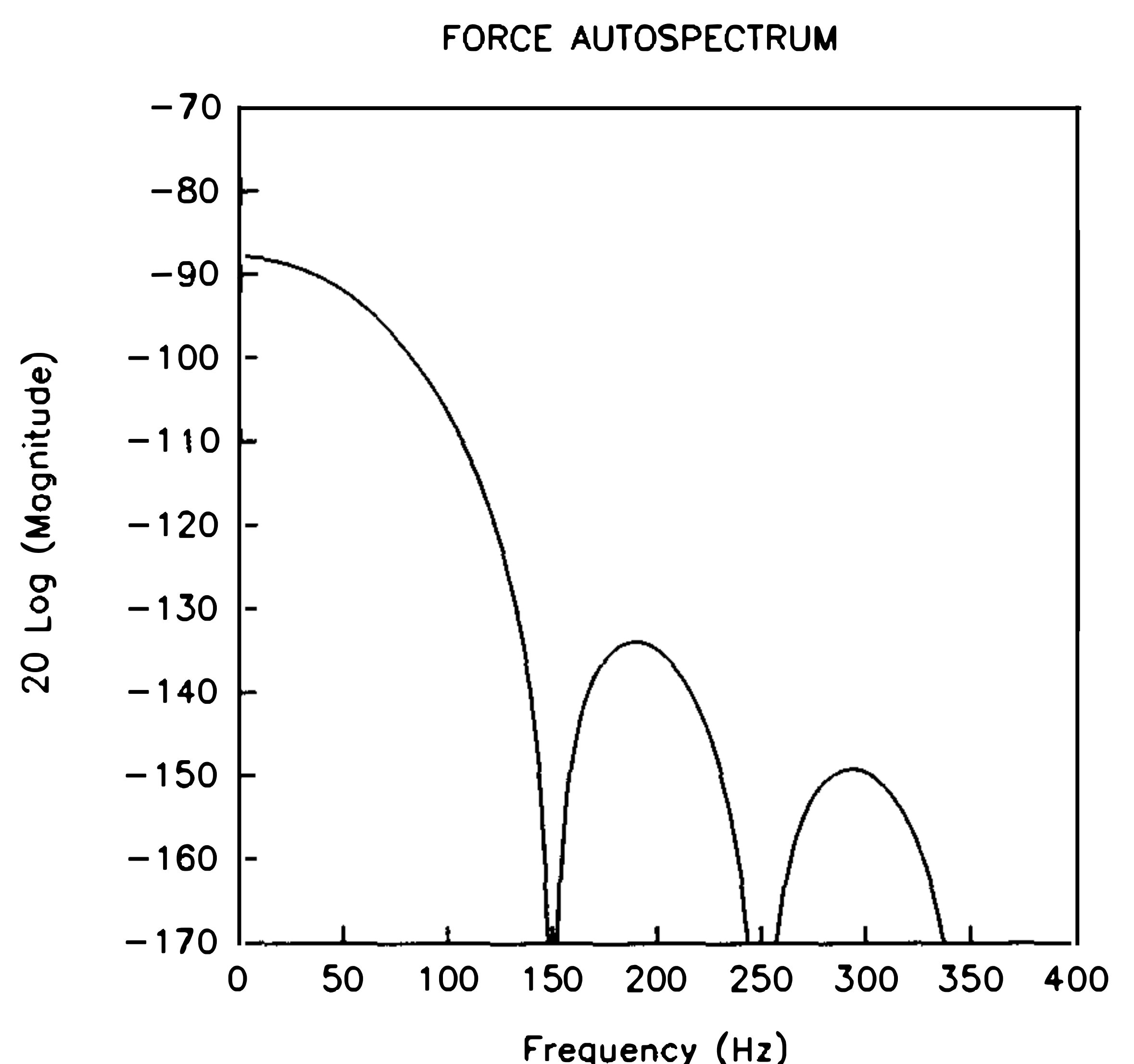
Fig. 3 Auto spectrum for half sine impact force in Fig. 2a, T_1=0.01 (sec), F_0=1 (units of force)

$$\tau_{system} = \frac{1}{\zeta\omega_n} = \frac{1}{(0.01)(10[2\pi])} = 1.59 \text{ sec} \tag{16}$$

Assume the system is excited by an impulse of duration 10.0 ms with a force level arbitrarily set to 1 N. The time response signals are illustrated in Fig. 2. The entire impulse signal is easily captured, while the light damping causes the system response to decay slowly requiring over 15 seconds to elapse before the amplitude is negligible. Typically a capture duration of this length is impractical. To examine the effect of truncation on the spectral functions, consider the application of Eqs. (13) and (14) to this data with two different capture durations.

First, consider a capture duration equal to 1/2 system time constant, T=0.795 sec. Figure 2 illustrates the truncation of the vibration signal is significant. In practice, processing of signals truncated this severely would be judged unacceptable, however they will be used to accentuate the induced bias errors. The force spectrum is shown in Fig. 3 and is not subjected to any truncation effects because it is captured in it's entirety. The spectrum has a decaying $|sinc(f)|$ type characteristic. The first null occurs at frequency of $3/2T_1$ and subsequent nulls at integer multiples of $1/T_1$. Provided the test frequency range of interest is approximately less than $1/T_1$ (amplitude decreased by approximately 10 dB) a broadband force is produced. The resulting broadband force characteristic from an impulse is one of the most attractive features of impact excitation.

The theoretical (Eq. (15)) and estimated (Eq. (14)) frequency response functions are compared in Fig. 4. The estimated function exhibits a bias in both the magnitude and phase. The magnitude has the proper basic shape, however the truncation causes a $|sinc(f)|$ type function to be superimposed on the theoretical function. The values around the resonant frequency are most significantly affected. The characteristics of the peak are smeared into adjacent frequencies and the estimated phase oscillates around the theoretical value. The phase agrees well at the natural frequency, but deviates at surrounding frequencies.

Next consider a more realistic capture duration corresponding to 2 time constants of vibration data, T=2/$\zeta\omega_n$=3.18 sec. The force spectrum remains unchanged from Fig. 3. The estimated frequency response
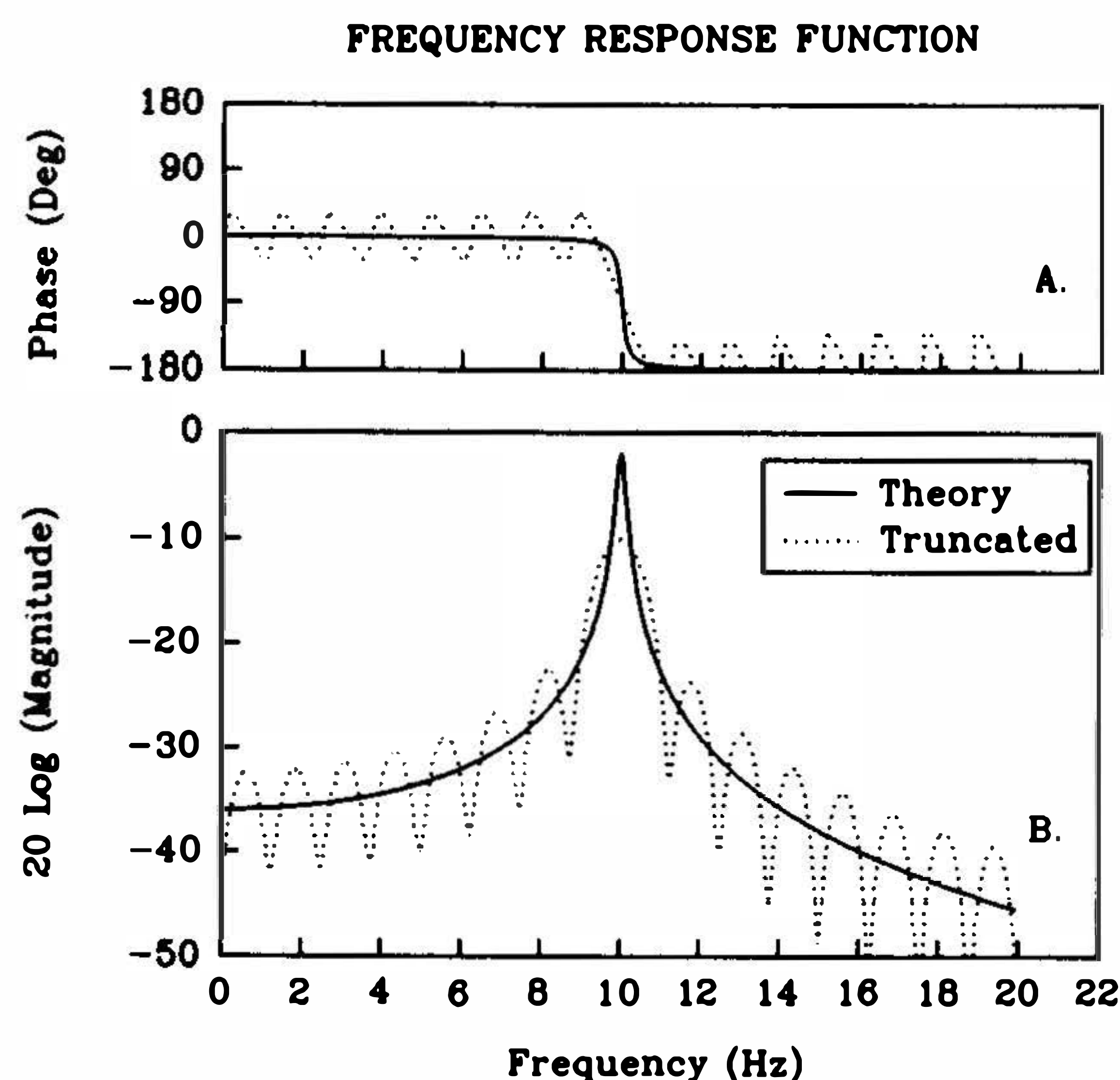
Fig. 4 Frequency response function estimated with 1/2 of a time constant of data (f_n=10Hz, ζ=0.01) - [Theoretical ———, Estimated ——]
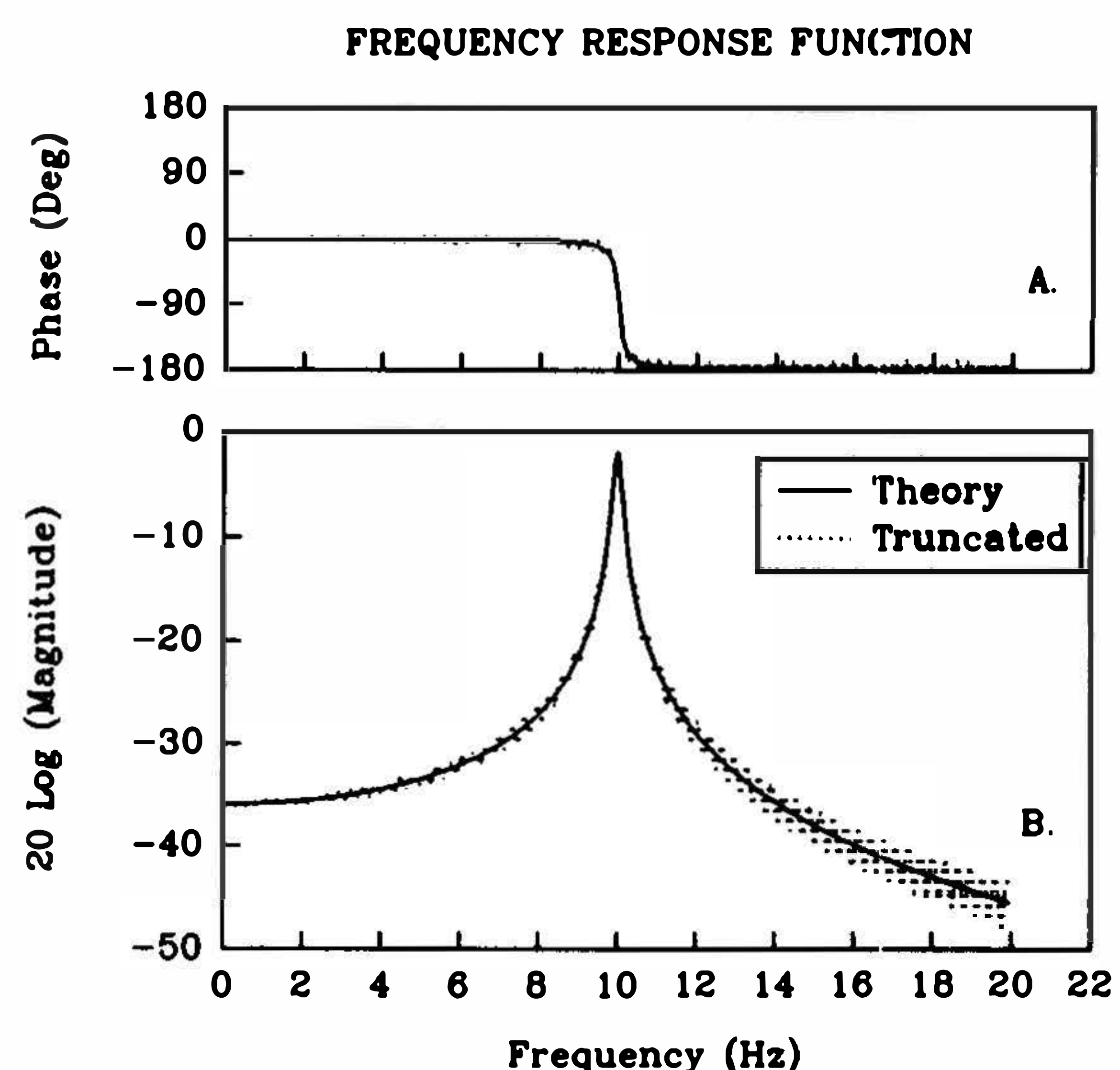
Fig. 5 Frequency response function estimated with 2 time constants of vibration data (f_n=10 Hz, ζ=0.01). [Theoretical ———, Estimated ——]

function is shown in Fig. 5. The magnitude shows a $|sinc(f)|$ as before, however the function is closer to the theoretical. The correlation is significantly improved around the resonant frequency in both magnitude and phase.

By comparing Figs. 4 and 5 it is apparent that the capture time plays an important role in producing an accurate estimate of the frequency response function. Establishing a relationship between the capture duration and the induced bias error would provide a criteria to maintain the truncation error below an acceptable level. Examination of Figs. 4 and 5 shows that the error is significant at frequencies near the natural frequency. Furthermore, since data must be most accurate in this region for modal analysis applications, it is logical that any analysis concentrate on these frequencies. The frequency response function magnitude error at the natural frequency was evaluated with increasing data capture durations with the results shown in Fig. 6. The percent error in the frequency response function magnitude (at f_n) is presented versus the number of time constants of data captured. The relationship shows large error values exist for high degrees of truncation. As the amount of data captured increases to around five time constants, the error decreases dramatically. If at least 6 time constants of data are captured the estimated response function magnitude is within 0.5% and the phase is within 0.25 degrees of the respective theoretical values. Both of these values are valid for a baseband frequency range equal to twice the system's natural frequency (i.e., $f_{analysis}=2f_n$).

The number of time constants of data captured can be directly related to the analysis frequency selected by an analyzer operator. The six time constant capture criteria implies the inequality

$$\zeta 2\pi f_n T > 6 \tag{17}$$

The capture period T is related to the sampling time via the FFT block size ($T=N\Delta t$) or the sampling frequency ($f_{sample}=1/\Delta t$).

$$\zeta 2\pi f_n N\Delta t = \frac{\zeta 2\pi f_n N}{f_{sample}} > 6 \tag{18}$$

In order to satisfy the Nyquist sampling criteria, the maximum frequency sampled, f_{sample}, must be at least twice the maximum desired frequency, $f_{analysis}$. In practice f_{sample} is usually more than twice $f_{analysis}$. Given
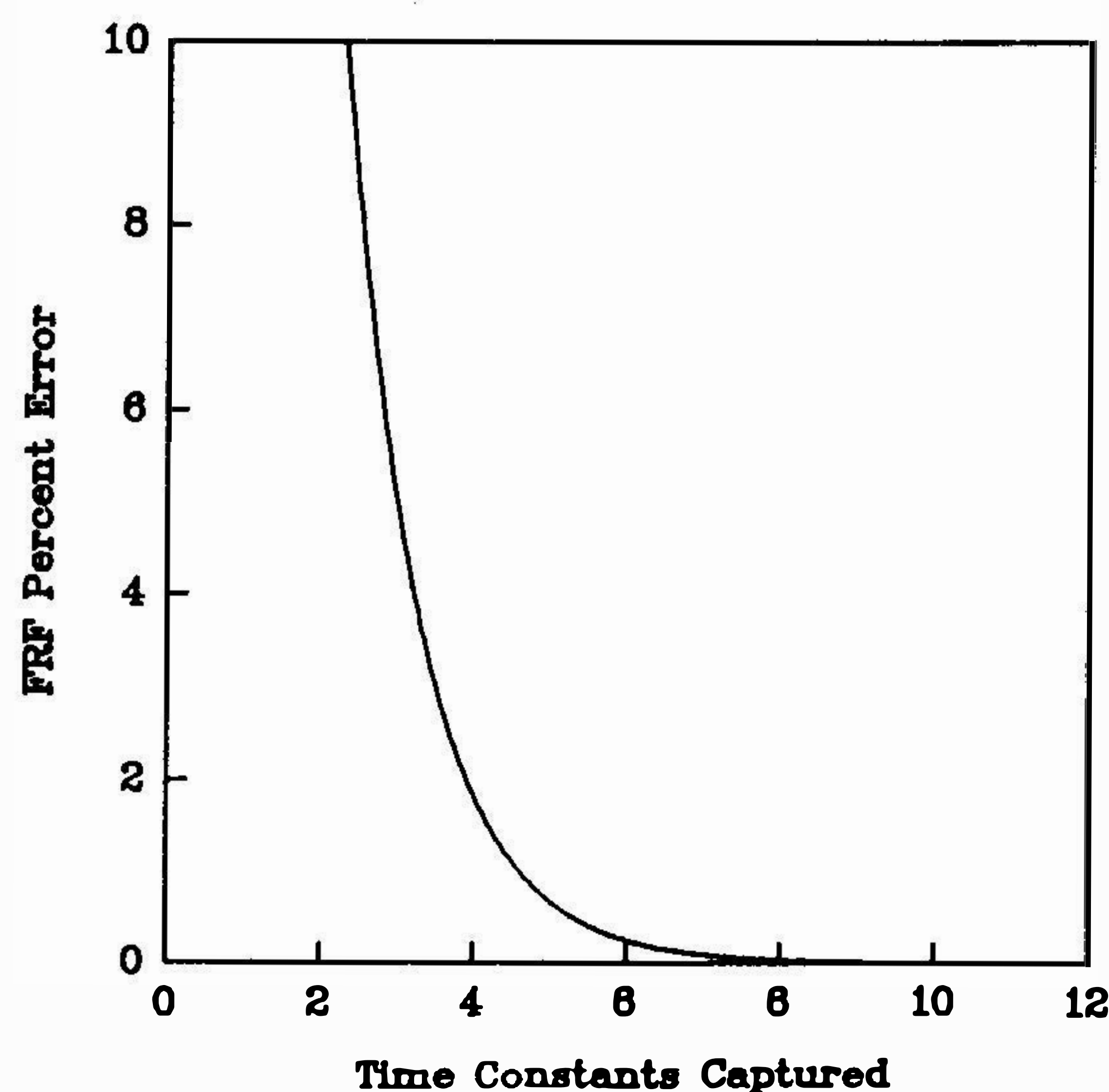

Fig. 6   Percent error of the frequency response function magnitude at the natural frequency of a single degree-of-freedom system versus time constants of response data captured.

a 400 line FFT analyzer with 1024 point blocksize, the analysis frequency is related to the sampling frequency by

$$f_{sample} = \frac{N}{400} f_{analysis} = \frac{1024}{400} f_{analysis} \tag{19}$$

Substituting Eq. (19) into Eq. (18) yields

$$\frac{\zeta 2\pi f_n N}{\frac{N}{400} f_{analysis}} = \zeta 2\pi 400 \frac{f_n}{f_{analysis}} > 6 \tag{20}$$

Solving Eq. (20) for the damping ratio yields

$$\zeta > \frac{6}{(2\pi)400} \frac{f_{analysis}}{f_n} = 0.0024 \frac{f_{analysis}}{f_n} \tag{21}$$

Eq. (21) shows that the ratio of the analysis frequency to the system natural frequency determines the damping ratio necessary to satisfy the data capture criteria. For a frequency ratio ($f_{analysis}/f_n$) of 10, a damping ratio of at least 0.024 is required to capture at least six time constants of data. If the frequency ratio is 1.1 only a damping ratio of 0.0027 is required to meet the criteria.

Since the force impulse is entirely captured in the data window, the force autospectrum is free from any truncation bias error. The bias error in the frequency response function is solely induced by an insufficient capture of the slowly decaying vibration signal. Conditions for the error to be significant exist when the damping is light and the ratio of the selected analyzer analysis frequency range to the system natural frequency is large.

When the data capture criteria developed in Section 3 can not be satisfied an exponential data capture window may be applied to the system response to force it to decay faster. An exponential window is defined by

$$W_{EXP}(t) = \begin{cases} e^{-\frac{t}{\tau_w}} & 0 \leq t \leq T \\ 0 & T < t \end{cases} \tag{22}$$

where the time constant, τ_w, is selected by the operator. Often, τ_w is selected so the vibration is forced to decay to 2% of its amplitude within the capture duration

$$\tau_w = -\frac{T}{\ln(0.02)} = 0.2556T \ \ (sec) \tag{23}$$

This window is referred to as the 2% exponential window.

Fourier transforms of the windowed samples are affected by this windowing operation. Consider Eq. (6), with the boxcar window replaced by the 2% exponential window.

$$X(\omega,T) = \int_{-\infty}^{\infty} W_{EXP}(t)x(t)e^{-i\omega t}dt$$

$$= \int_{0}^{T} e^{-\frac{t}{0.2556T}} x(t)e^{-i\omega t}dt \tag{24}$$

Substitution of the single degree-of-freedom system vibration time response, Eq. (9), into Eq. (24) provides a means to evaluate the frequency domain effects of the exponential windowing operation. An alternative to performing the resulting integration involves the variable substitution shown in Eq. (25) into the unwindowed Fourier transform [3].

$$\omega = \omega - \frac{i}{\tau_w} \tag{25}$$

Equation (25) was applied to Eq. (9) in subsequent numerical evaluations of exponential windowing operations.

The windowing effects on the Fourier transforms can also be viewed in the frequency domain as a convolution of the actual signal transform with the window transform [6].

$$Y(\omega,T) = \int_{-\infty}^{\infty} W(\alpha)Y(\omega - \alpha)d\alpha \tag{26}$$

where $Y(\alpha)$ is the Fourier transform of the system response, $W(\alpha)$ is the window Fourier transform and α is an integration variable. Equation (26) implies the spectral effects of exponential windowing can be examined with the respective window Fourier transforms. Figure 7 compares the Fourier transform magnitudes of a boxcar window (Eq. (5)), and an 2% exponential window (Eq. (22)) both with a duration T (τ_w=0.2556T). The figure shows the characteristic $|sinc(f)|$ leakage lobes of the boxcar window with nulls occurring at frequencies equal to integer multiples of 1/T. The 2% exponential window has a exponential
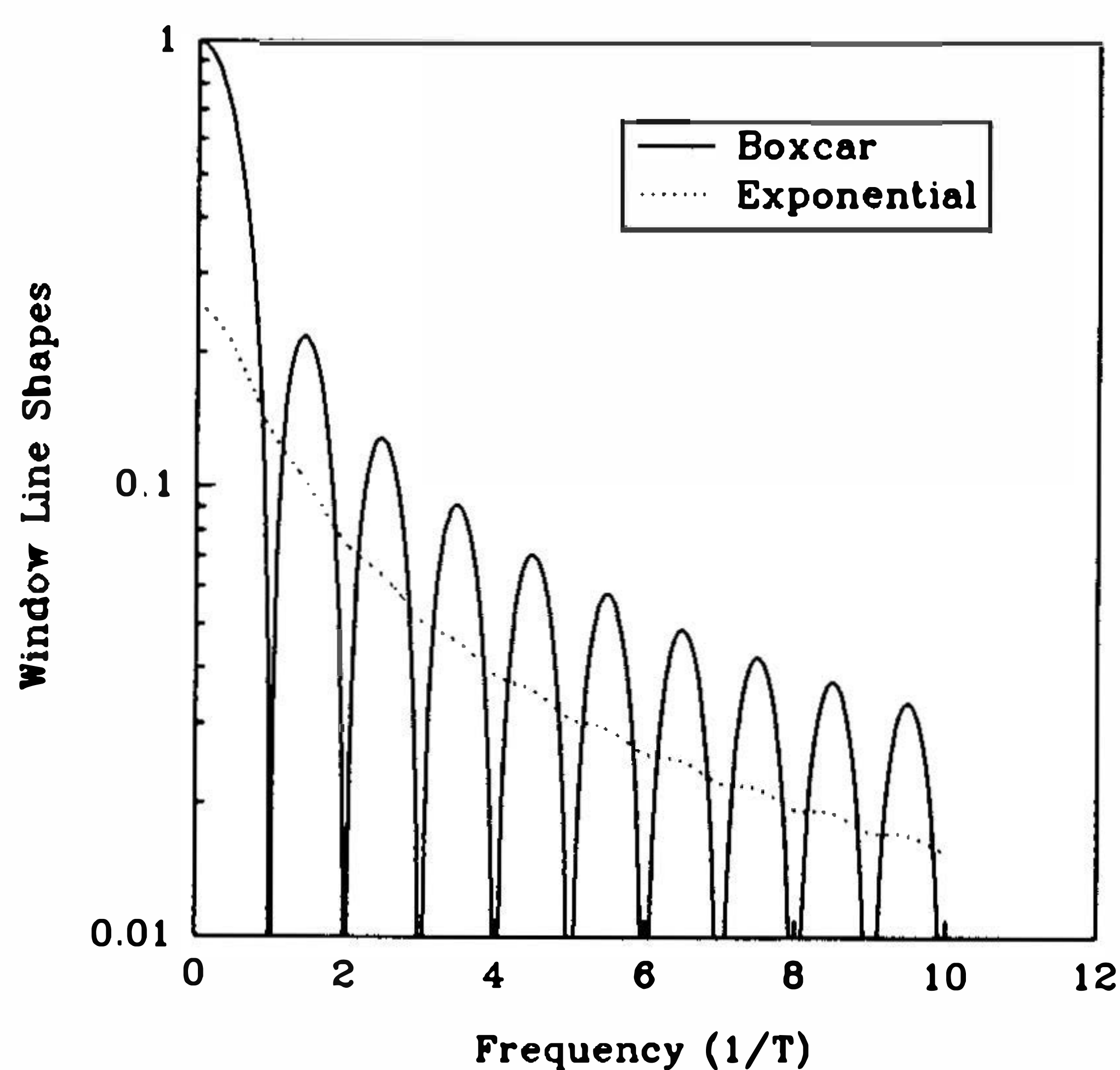
Fig. 7   Fourier transform magnitudes of a boxcar (duration=T) and an exponential window (duration=T, τ=T/ ln{0.02}).

type behavior with a small ripple at the higher frequencies and similar roll off characteristics.

The use of an exponential window to reduce truncation of a slowly decaying vibration signal raises several questions. First, to what level may the original signal be truncated and artificially forced to decay while not coloring the spectral information to preclude accurate modal parameter extraction? Secondly, because of the severe leakage characteristics of the exponential window, can the responses of closely spaced modes be obscured sufficiently such that they can not be separated?

(a) Truncation Effects in Exponentially Windowed Data

The truncation/windowing question was investigated numerically with the same single degree-of-freedom system (f_n=10 Hz, ζ=0.01) in Section 3. The system frequency response function was estimated for different capture times selected to accentuate the windowing effects. Another set of responses was generated by applying an exponential window to the truncated responses. The modal parameters were then extracted from each of the respective FRFs using a complex exponential modal extraction algorithm. The modal parameters extracted are the natural frequency (f_n), the damping ratio (ζ), and the modal residue (R). The estimated parameters were then compared to the theoretical values.

Figure 2 illustrates the half sine force pulse and the corresponding system response with a capture time of 3.18 seconds (two system time constants). The same response with a 2% exponential window applied is shown in Fig. 8. The decay envelope of the response includes both the inherent system and artificial window decay behavior. The more rapid decay of the exponentially windowed vibration response is apparent. The frequency response function estimated with a 3.18 second boxcar window is shown in Fig. 5 and the exponential windowed data results in Fig. 9. The windowing smoothed the $|sinc(f)|$ effect across the entire frequency range and reduced the magnitude around the resonance by about 10 dB.

To further accentuate the windowing/truncation effect the capture time was decreased to one system time constant, 1.59 sec. The frequency response function estimated with a 2% exponential applied to the vibration response signal is shown in Fig. 10. The amplitude at the resonance relative to the theoretical value decreases by about 16 dB. The phase at the natural frequency is identical but shows a lower gradient slope, indicative of the artificial damping induced by the window.
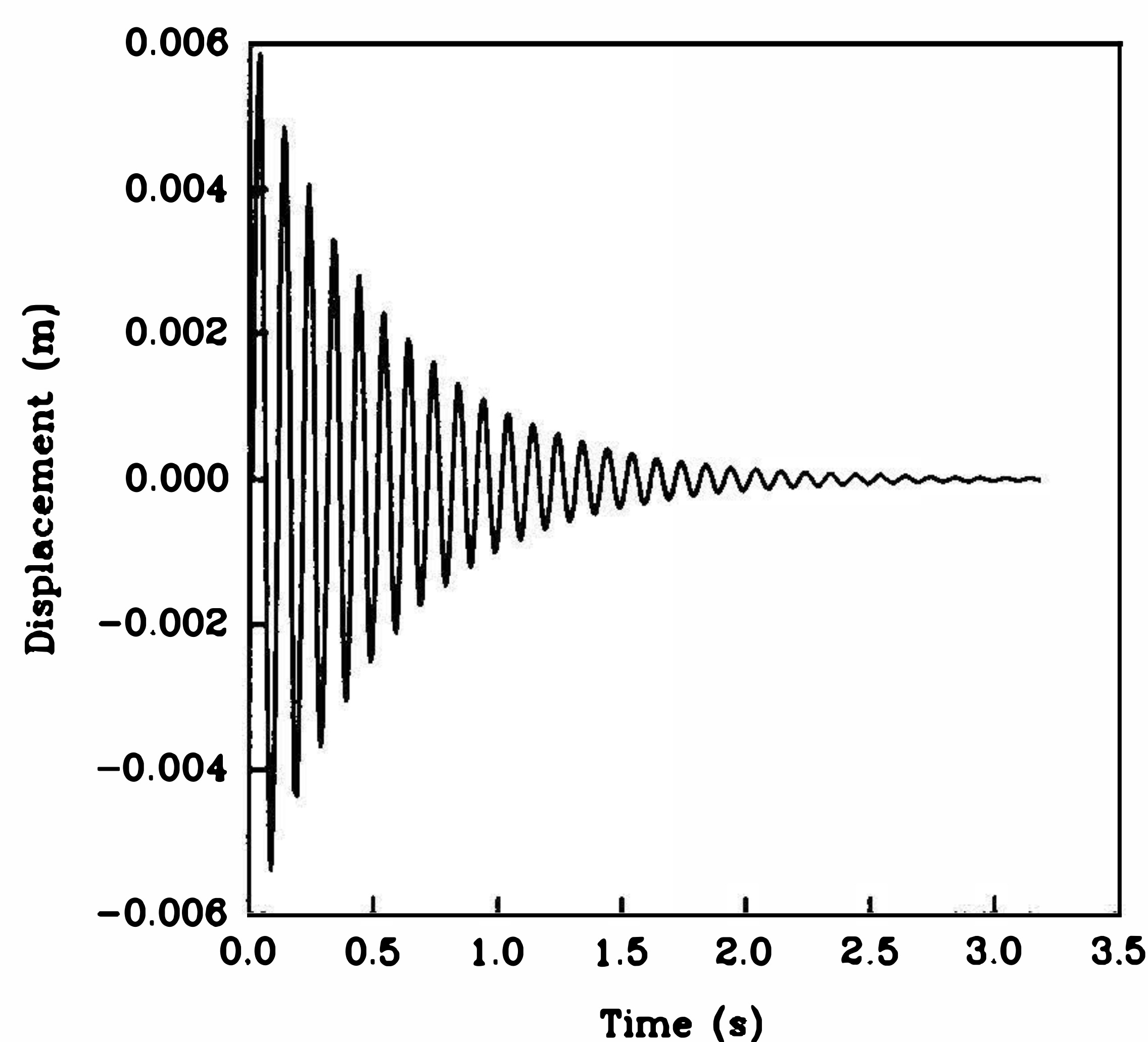

Fig. 8   Two time constants of vibraton response of a single degree-of-freedom (f_n=10 Hz, ζ=0.01) to the impact force in Fig. 2 after the applicaton of a 2% exponential window.
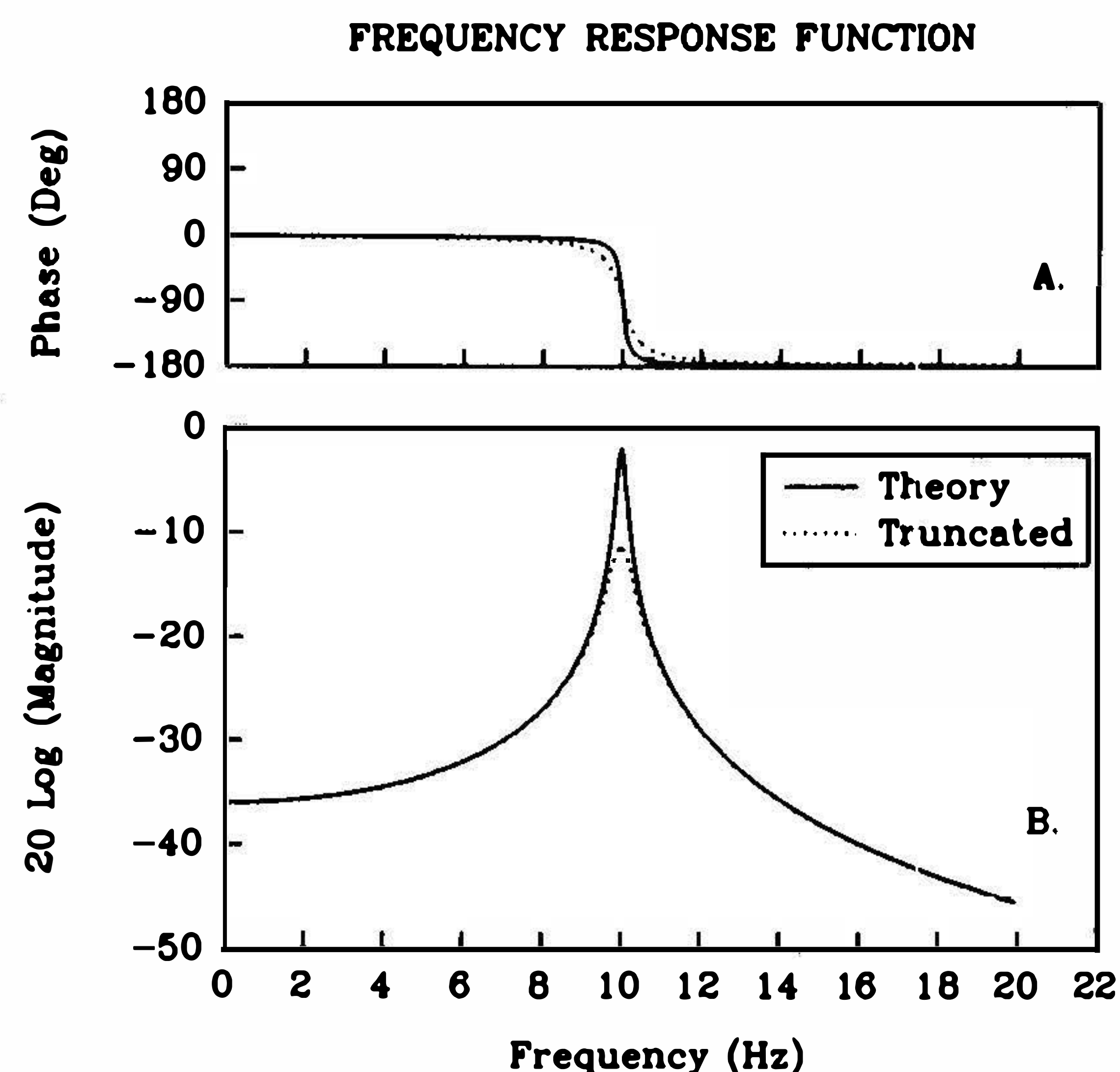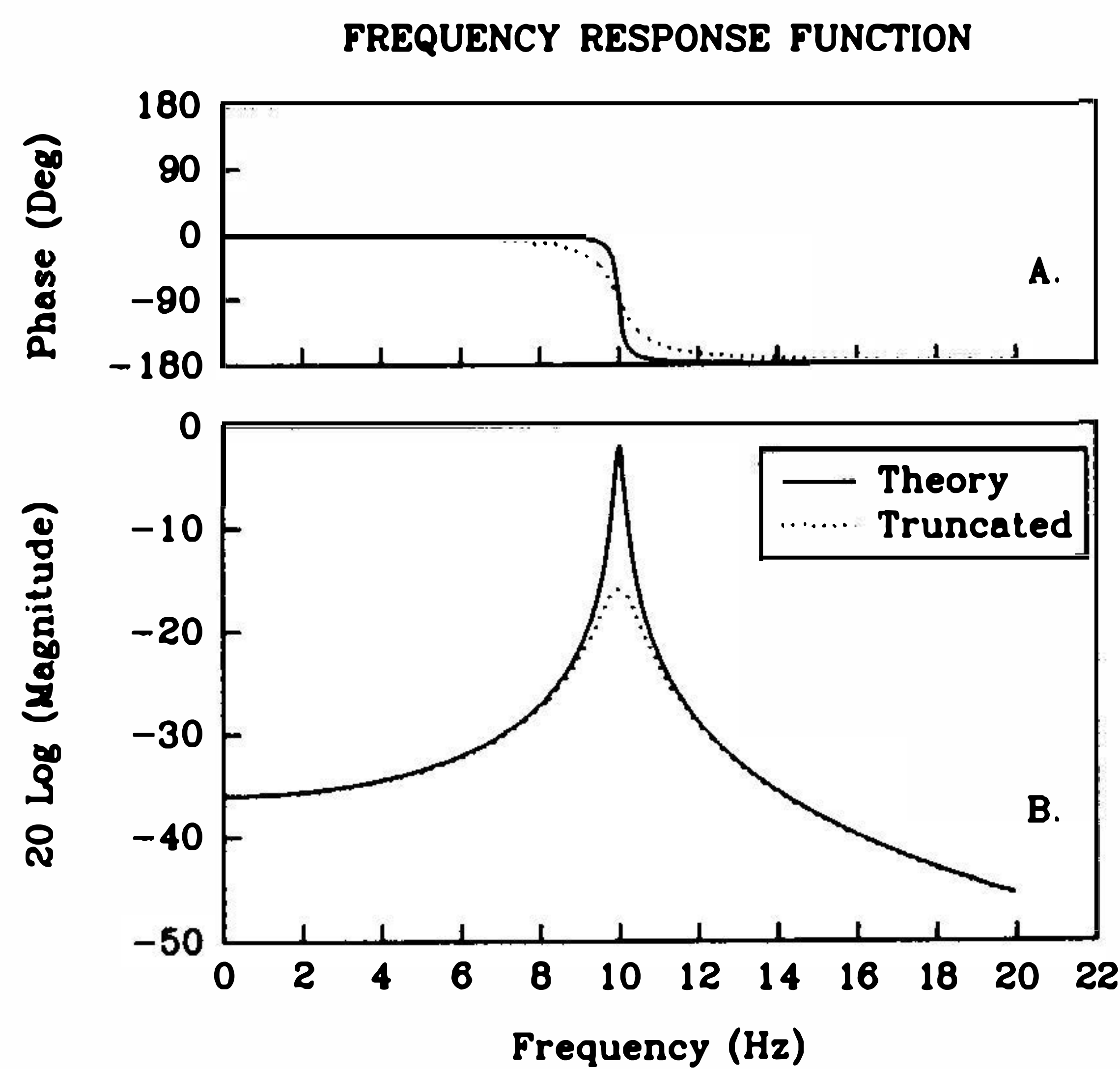
Fig. 9   Frequency response function estimated with 2 time constants of data with a 2% exponential window (f_n=10 Hz, ζ=0.01). [Theoretical ——— , Estimated——]

Clearly, these two examples show that the estimated FRFs differ from theory. The question remaining is, how much does this affect the modal parameter extracted from these functions? A complex exponential routine was used to estimate the modal parameters for four cases: windowed and unwindowed data with capture durations of one and two time constants respectively. The results are listed in Table 1. Resonance frequencies extracted for all the cases are within 1% of the theoretical value of 10 Hz. The damping values for the unwindowed FRFs, cases 1 and 3, are also reasonable compared to the theoretical value of 0.01. The damping is slightly higher for case 1 (1 time constant) than that for case 3 (2 time constants) as a result of leakage. The damping values for the windowed FRFs, cases 2 and 4, are at least three times larger than the theoretical value. This is caused by artificial damping induced by the exponential window. A corrected damping value can be calculated with knowledge of the exponential window time constant [3]

$$\zeta_{cor} = \zeta - \frac{1}{\tau_w \omega_n}$$

(27)

Fig. 10 Frequency response functions estimated with 1 time constant of data with a 2% exponential window (f_n=100 Hz, ζ=0.01). [Theoretical ———, Estimated ----]
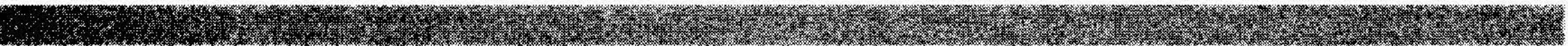
TABLE 1 EXTRACTED MODAL PARAMETERS FROM A SDOF SYSTEM IMPULSE TEST WITH BOXCAR AND EXPONENTIAL WINDOWS.

Case #	Window Type	Mode #	f_n (Hz)	ζ	ζ_{corr}	R	T (s)	τ (s)
Theory	None	1	10.00	0.0100	-	1.00	-	-
1	Boxcar	1	9.95	0.0133	-	1.01	1.59	-
2	2% Exp	1	10.00	0.0495	0.0106	0.99	1.59	0.409
3	Boxcar	1	10.00	0.0104	-	1.00	3.18	-
4	2% Exp	1	10.00	0.0299	0.0104	1.00	3.18	0.818

Applying Eq. (27) to the damping ratio estimates obtained from the windowed data produces corrected values within 1% of the theoretical. All the modal residue (R) values are within 1% of the true value.

In practice, difficulties can be encountered when trying to utilize Eq. (27) with actual experimental data. The numerical values of the equations' terms can be small which can induce large errors in the corrected damping ratio if the original damping estimate is inaccurate. Therefore, use of Eq. (27) must be handled carefully.

(b) Exponential Windowing with Closely Spaced Modes

Another question concerning exponential windows is the ability to distinguish closely spaced modes because of increased leakage. Due to the number of parameters and complexity of the relationships it is difficult to completely evaluate the problem. Therefore, the problem will be illustrated with several numerical examples where the effects are pronounced. The examples will describe the results from a two DOF system with close natural frequencies. To generate the system FRFs, decoupled modal coordinates are used allowing two SDOF FRFs to be superimposed with Eqs. (8), (9), (14) and (25).

The first mode was specified at 10 Hz with $\zeta=0.01$ and R=1.0 while the second mode was at 12 Hz with $\zeta=0.005$ and R=10.0.The theoretical FRF is compared to one estimated by a boxcar window (T=3.18 seconds: two time constants of data for the 10 Hz system, frequency resolution = 0.05 Hz) in Fig. 11. A

TABLE 2   EXTRACTED MODAL PARAMETERS FROM A TWO DOF SYSTEM IMPULSE TEST WITH BOXCAR AND EXPONENTIAL WINDOWS.

Case #	Window Type	Mode #	f_n (Hz)	ζ	ζ_{corr}	R	T (s)	τ (s)	Δf (Hz)
Theory	None	1	10.00	0.0100	-	1.00	-	-	0.05
1	Boxcar	1	9.89	0.0642	-	1.56	1.59	-	0.05
2	2% Exp	1	10.03	0.0501	00110	0.987	1.59	0.409	0.05
3	Boxcar	1	10.01	0.0106	-	1.00	1.59	-	0.25
4	2% Exp	1	10.03	0.0490	0.0100	0.978	1.59	0.409	0.25
5	Boxcar	1	10.03	0.0231	-	1.40	3.18	-	0.05
6	2% Exp	1	10.02	0.0302	0.0107	1.00	3.18	0.818	0.05
7	Boxcar	1	10.01	0.0104	-	0.961	3.18	-	0.25
8	2% Exp	1	10.02	0.0299	0.0104	0.995	3.18	0.818	0.25
Theory	None	2	12.00	0.0050	-	10.00	-	-	0.05
9	Boxcar	2	12.00	0.0099	-	11.10	1.59	-	0.05
10	2% Exp	2	12.00	0.0376	0.0052	9.90	1.59	0.409	0.05
11	Boxcar	2	12.00	0.0051	-	5.80	1.59	-	0.25
12	2% Exp	2	12.00	0.0370	0.0046	9.90	1.59	0.409	0.25
13	Boxcar	2	12.00	0.0070	-	10.60	3.18	-	0.05
14	2% Exp	2	12.00	0.0214	0.0052	9.97	3.18	0.818	0.05
15	Boxcar	2	12.00	0.0051	-	7.54	3.18	-	0.25
16	2% Exp	2	12.00	0.0213	0.0051	9.95	3.18	0.818	0.25

similar comparison is presented in Fig. 12 with the frequency resolution increased by a factor of five to 0.25 Hz. The leakage effects are more pronounced. Figure 13 shows the FRF estimated with a frequency resolution of 0.25 Hz when an 2% exponential window is applied to the vibration response signal. The peak amplitudes are lower than theory and they tend to blend into one another as a result of the exponential window leakage.

To examine these effects on the modal parameters extracted under these conditions a complex exponential algorithm was applied to 16 different cases with the following parameters varied: 1) T - the capture time; 2) Δf - the frequency resolution; 3) window type - boxcar or 2% exponential. The results are summarized in Table 2. The modal extraction routine predicts the resonance frequencies for all of the cases
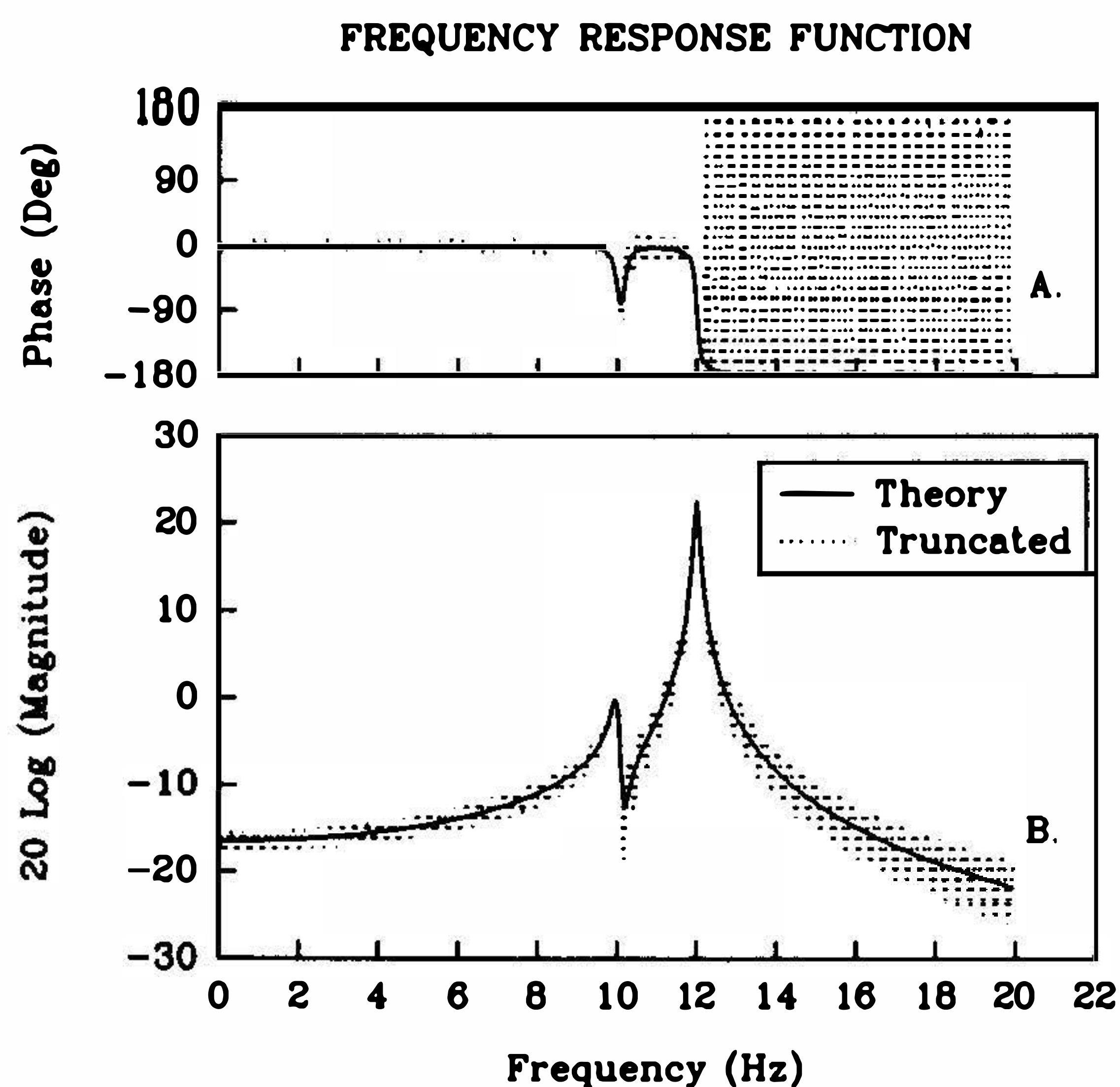
Fig. 11   Two DOF FRF estimated with 2 time constants of data of mode 1 (f_n=10 Hz, ζ=0.01) and Δf=0.05 Hz. [Theoretical ——, Estimated -----]
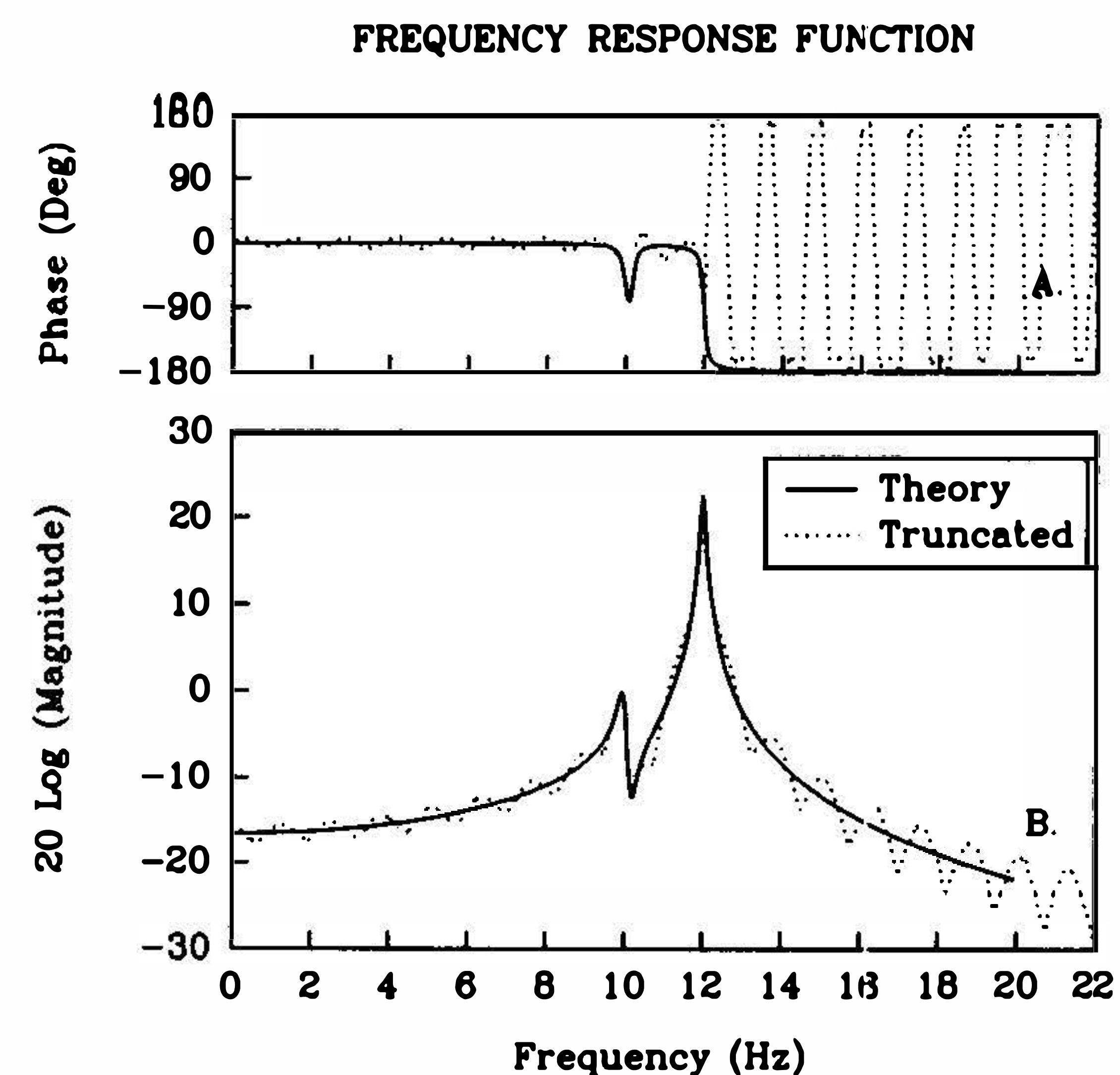
Fig. 12   Two DOF FRF estimated with 2 time constants of data of mode 1 (f_n=10 Hz, ζ=0.01, and Δf=0.25Hz. [Theoretical ——, Estimated -----]
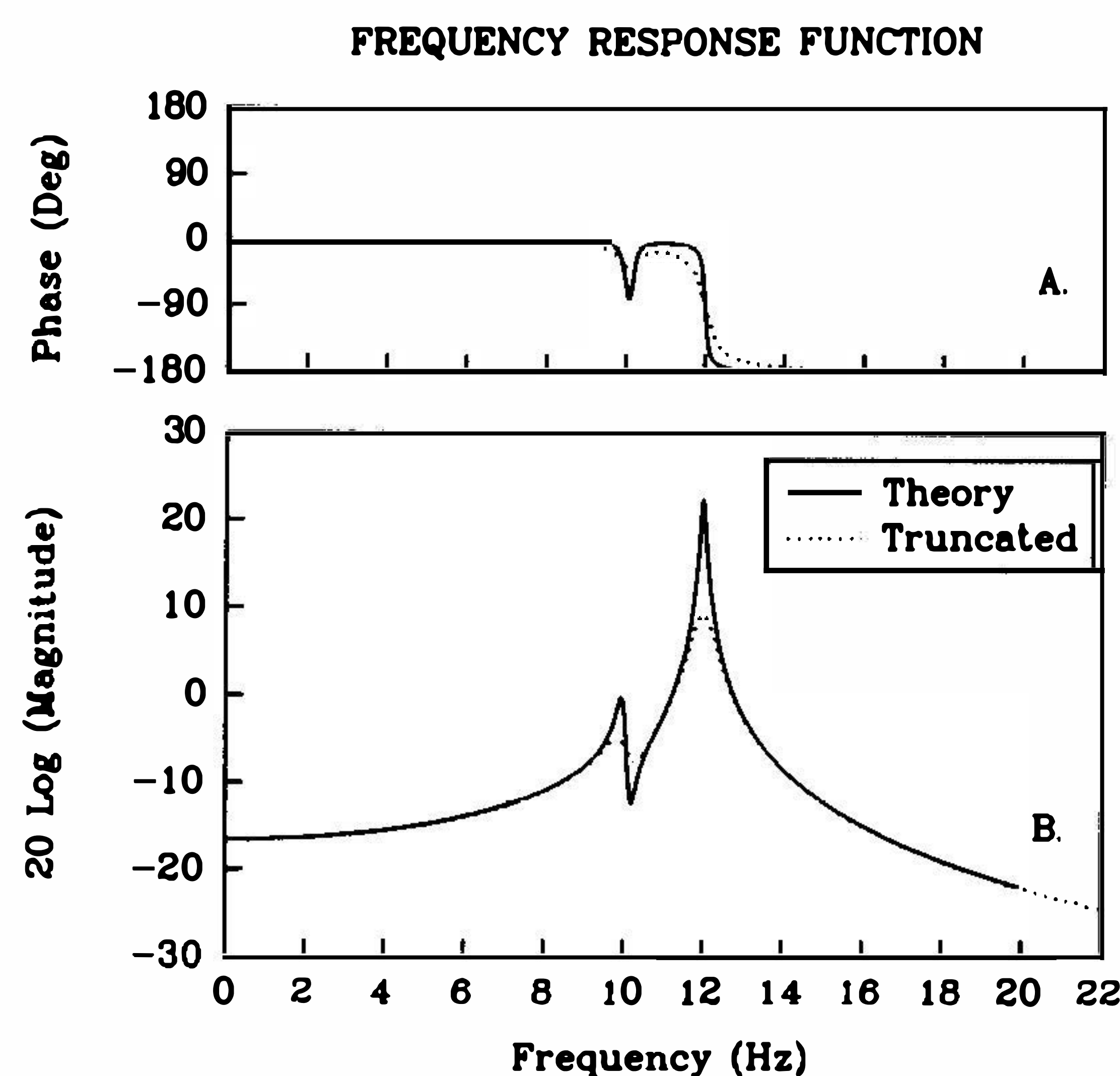
Fig. 13   Two DOF FRF estimated with 2 time constants of data of mode 1 (f_n=10 Hz, ζ=0.01) with a 2% exponential window and Δf=0.25 Hz. [Theoretical ——, Estimated -----]

within 2% of the theoretical values. The damping values for the truncated unwindowed data (cases 1,5,9,13) differ from the theoretical values significantly. The frequency resolution of these FRFs was 0.05 Hz. The damping values for the truncated unwindowed data (cases 3,7,11,15) all compare to within 1% of the true values. These cases have a frequency resolution of $\Delta f = 0.25$ Hz. The corrected damping values for the windowed data (cases 2,4,6,8,10,12,14,16) are all within 1% of the true values. In general, the modal residues for the windowed data (cases 2,4,6,8,10,12,14,16) all compare well (within 1%) with the true values for Δf equal to both 0.05 and 0.25 Hz. The modal residues for the boxcar windowed data with $\Delta f = 0.25$ Hz (cases 3,7,11,15), differ from the true values by as much as 50%.

Some observations can be made from the data presented in Table 2. The modal extraction routine predicts the resonance frequencies accurately for all the cases of truncation, windowing, and frequency resolution studied. Modal residues and corrected damping values for all exponentially windowed data are accurate. The two different frequency resolutions studied did not significantly influence the modal parameter extraction. Both resolutions were fine enough to adequately separate the closely spaced modes.

Although difficult to make explicit statements about signal truncation and in relation to the application of exponential windows, some general observations can be made. This analysis has demonstrated that spectral functions estimated from impulse tests are affected by the windowing/truncation of slowly decaying vibration response data. However, the results also indicated that modal parameters estimated from this spectral data are rather robust and not significantly affected by the truncation. This apparent contradiction is answered by realizing that the complex exponential modal extraction routine used in this study was able to recover accurate modal parameters from even spectral data which is visibly biased. Therefore, the agreement obtained in this study must be attributed to the capabilities of the modal extraction routine and not the robust nature of the data. Since similar results may not be obtainable with other algorithms, it is important to estimate the most accurate FRFs possible. Therefore, it is better to increase the capture duration to capture at least six time constants of data rather than apply an exponential window. If the window must be applied one must be aware of consequences associated with windowing, specifically: 1) coloring and leakage of the spectra induced by the convolution of the window transform; 2) possible numerical difficulties in back calculating the true damping ratio.

When physically impacting a structure with an instrumented hammer, a so called "double hit" occasionally occurs. Under this condition two pulses are applied to the structure, as illustrated in Fig. 14. The cause of the second impact can be from many sources related to the test procedures and system dynamics. A common occurrence is when the structure's response to the initial impact is rapid and large, such that a second contact is made with the hammer before it is fully retracted. Experimentalists usually reject "double hits" from entering the ensemble average set used to estimate the spectral quantities. These precautions are implemented because past experience shows that the quality of a FRF estimate is degraded when a double hit is included in the averaging ensemble.

To examine the double hit phenomena, consider the application of a double hit type force, as illustrated in Fig. 14, to the single degree-of-freedom system in Fig. 1. Since the system is linear, the expressions presented in Section 2 can be combined to produce both the temporal and spectral quantities. The vibration time response signal is

$$x(t) = x_1(t) + x_2(t - \gamma_0)u(t - \gamma_0) \tag{28}$$

where $x_1(t)$, and $x_2(t)$ are obtained from Eq. (2) with F_0 and T_1 equal to the respective characteristics of each of the two pulses, $u(t-\gamma_0)$ is a unit step function and γ_0 is the time delay between the impulses. Similarly, the Fourier transforms of the force and response can be evaluated by combining Eqs. (8) and (9) with the relationship

$$Y(\omega,T) = Y_1(\omega,T) + Y_2(\omega,T)e^{-i\omega\gamma_0} \tag{29}$$

where $Y(\omega,T)$ represents the respective characteristics from either the first or second impulse.
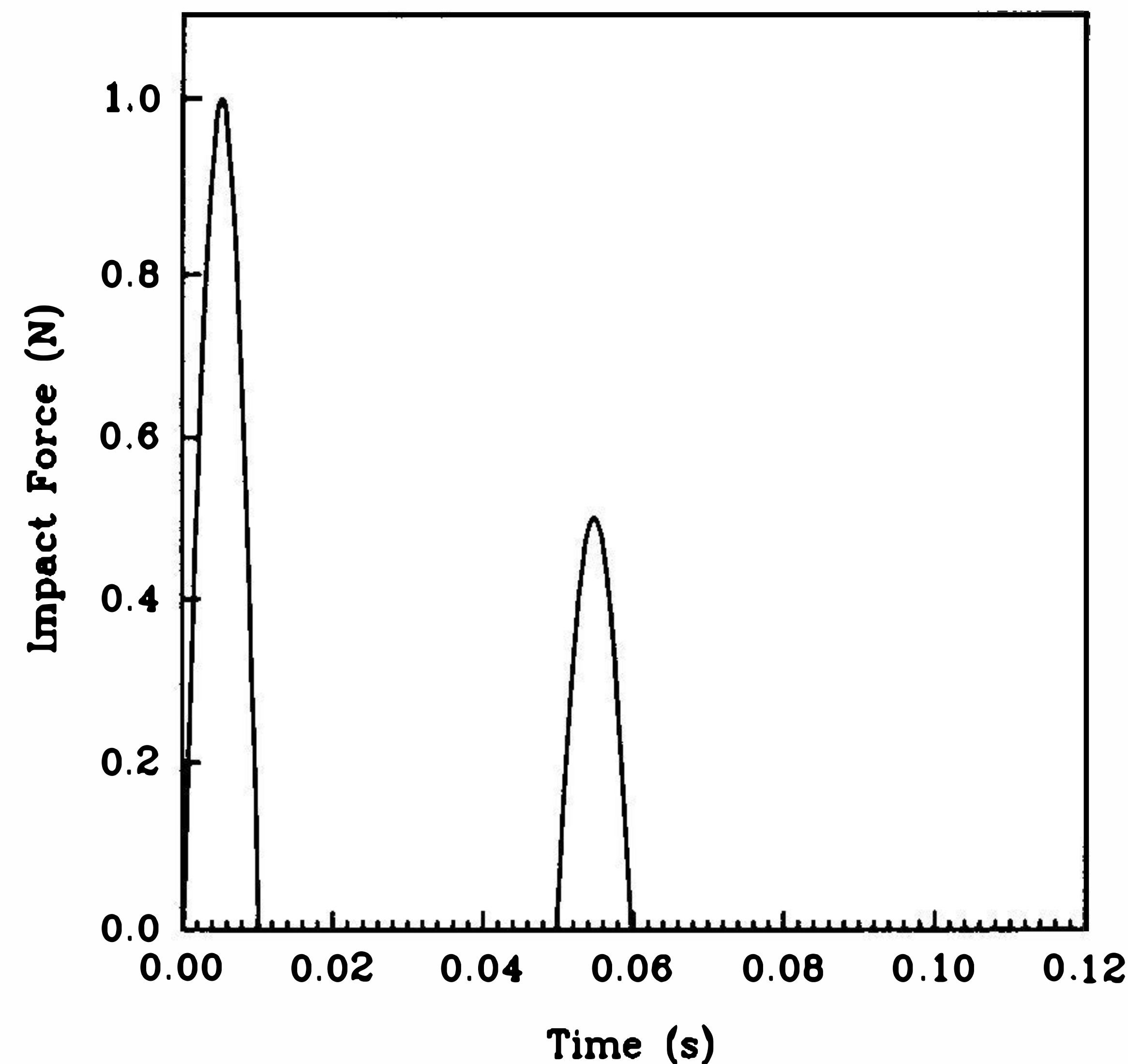
Fig. 14 Typical double hit experienced in a structural impact test
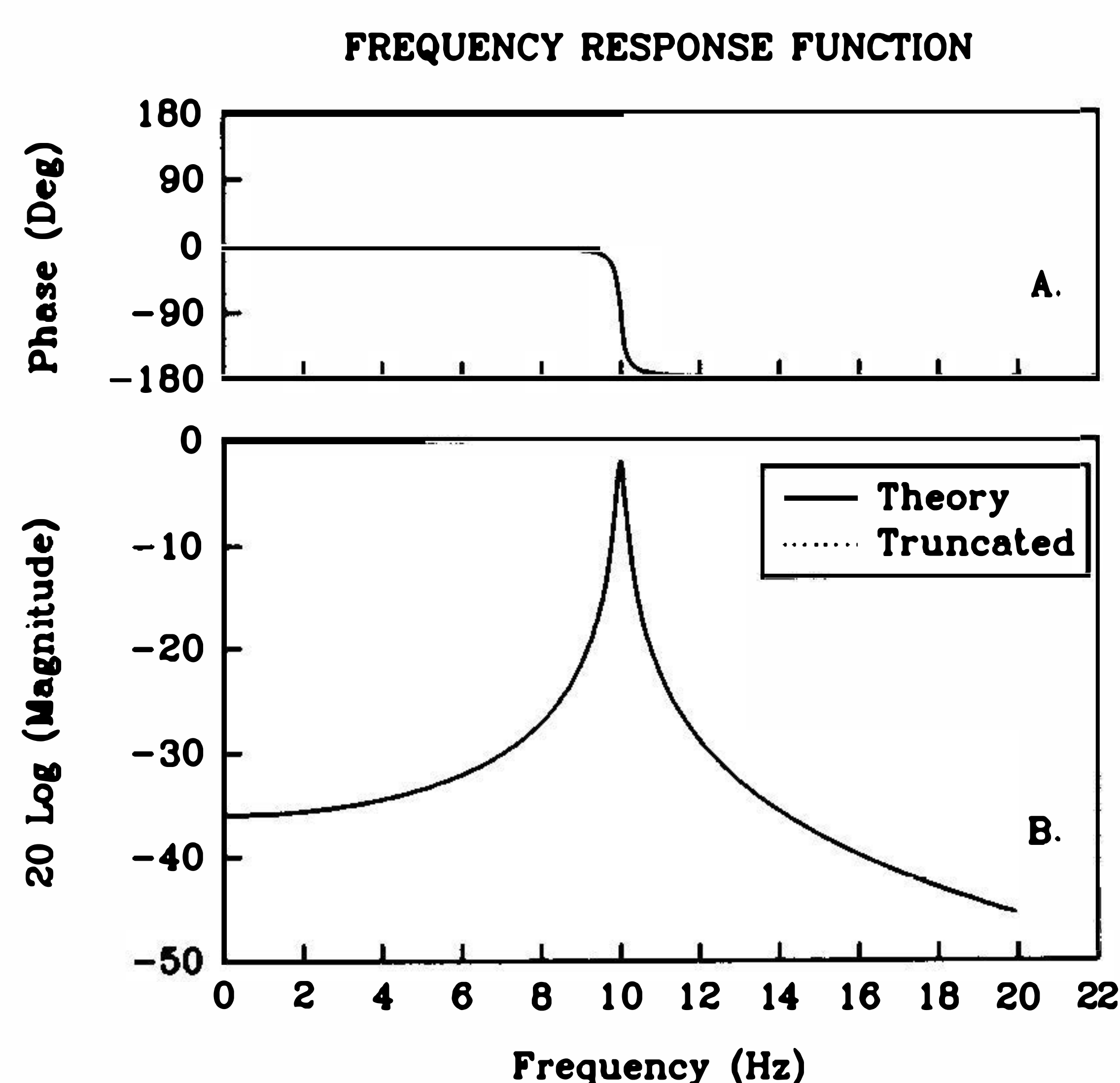
Fig. 15 FRF estimated with 6 time constants of data (f_n=10 Hz, ζ=0.01) excited by the double hit in Fig. 14. [Theoretical ———, Estimated ------]
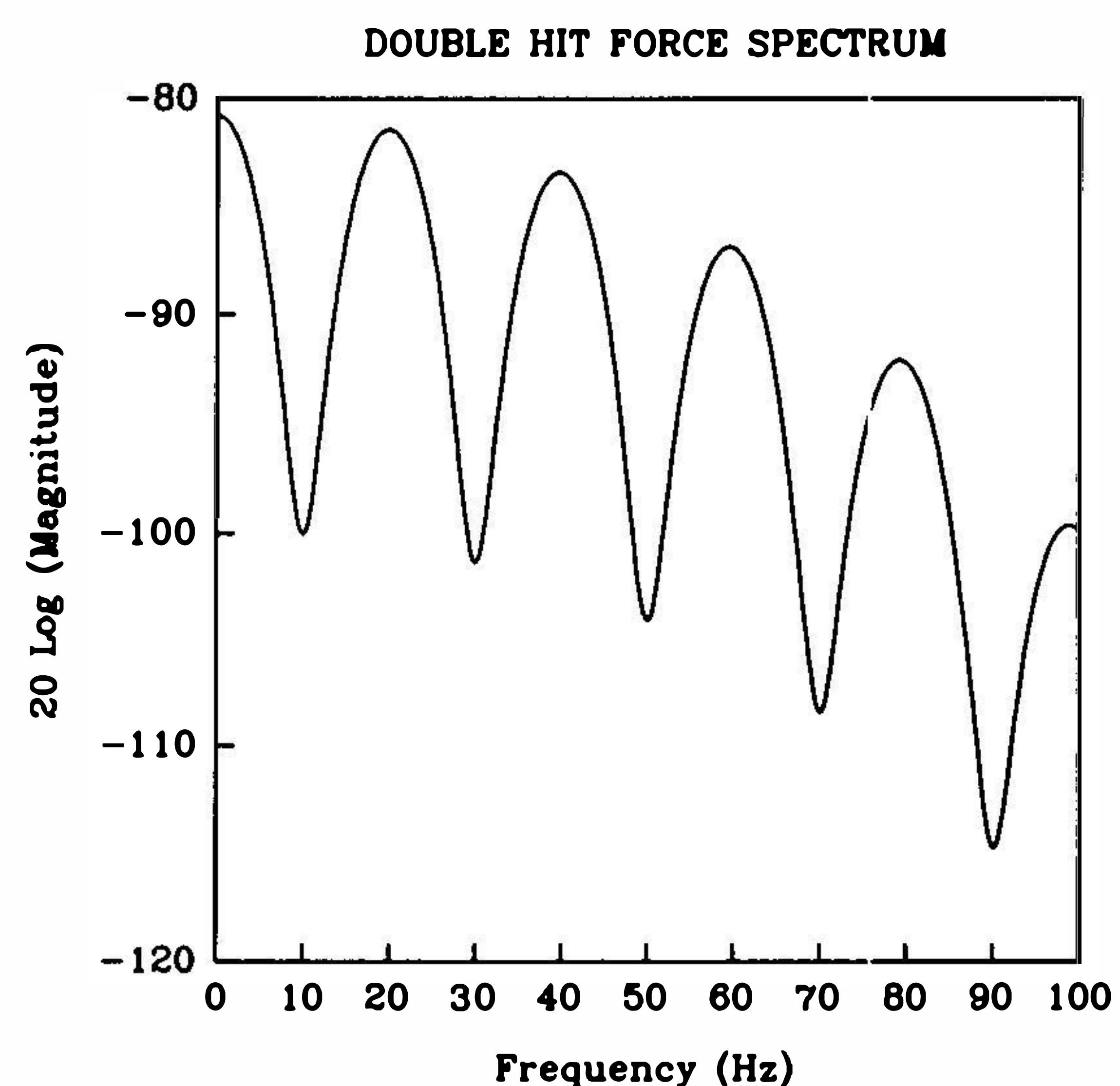
Fig. 16 Auto spectrum for a double hit half sine impact forces in Fig. 2a, T_1=0.01 (sec), F_1=1, F_2=0.5 (units of force) time delay = 0.05 sec.

Consider the force in Fig. 14 with $T_1=0.01$, $\gamma_0=0.05$ with the second impact magnitude half the first, applied to the single degree-of-freedom system used in the previous Sections ($f_n=10$, $\zeta=0.01$). The capture duration was set to six system time constants after the application of the second impulse ($T=6/\zeta\omega_n + \gamma_0$). The frequency response function calculated under this set of conditions is compared to the theoretical values in Fig. 15. There is no visible difference between the two curves (0.25% magnitude error at f_n), indicating an accurate estimation of the response function. This implies that provided that a sufficient length of response data is captured, at least six time constants after the second impact, accurate FRFs can be determined with double hit impacts.

The result in Fig. 15 should be readily expected from linear system theory but does not explain the apparent degradation the experimentalist experiences under similar conditions. Insight into the question is produced through examining the auto spectrum of the double hit in Fig. 16. The characteristics of this spectrum are significantly different from those produced when only a single impact is applied to a structure, Fig. 3. The single impact produced a broadband spectrum with the magnitude gently rolling off and eventually decreasing by approximately 10 dB at a frequency of $1/T_1$ or 100 Hz. On the contrary, the double hit spectrum is oscillatory in nature with amplitude differences of approximately 10 dB every 10 Hz. The single pulse spectrum decreases by less than 0.5 dB over the same frequency band. The double hit has greatly diminished a major characteristic of impact testing which makes it so attractive; a flat, broadbanded excitation.

The characteristics of the double hit force spectrum can be more readily examined by combining Eq. (8) with Eq. (29). When the only difference between the first and second impacts is the pulse magnitude, the Fourier transform of a double hit is given in Eq. (30)

$$F\left(\omega, T_1, \gamma\right) = F_1\left(\omega, T_1\right) + F_2\left(\omega, T_1\right)e^{-i\omega\gamma_0 > T_1}$$

$$= F_1\left(\omega, T_1\right)\left[1 + \frac{F_2}{F_1}e^{-i\omega\gamma_0}\right] \tag{30}$$

where F_1 and F_2 represent the respective peak amplitudes of the two impulses and $F_1(\omega,T_1)$ is the Fourier transform of the half sine pulse given in Eq. (8). Substituting Eq. (30) into Eq. (11) yields the autospectrum of a double impact

$$G_{FF}^T\left(\omega, T_1, \gamma_0\right) = G_{F_1F_1}^T\left(\omega, T_1\right)\left[1 + \left(\frac{F_2}{F_1}\right)^2 + 2\frac{F_2}{F_1}\cos\left(\omega\gamma_0\right)\right] \tag{31}$$

where the first term on the right side of the equation is the autospectrum if only the first impact was applied. Inspection of Eq. (31) shows parameters controlling the characteristics of the double hit spectrum. The oscillatory nature is controlled by the time between the impulses, the longer the delay, γ_0, the greater the number of oscillations. The amplitude differences are controlled by the relative magnitudes, F_2/F_1, of the two impulses. The greater the magnitude of the second impulse, the greater the magnitude differences in the spectrum.

From a theoretical standpoint, these spectral characteristics are immaterial and will not affect an FRF estimated with them, as demonstrated by the data in Fig.15. However, pragmatically they induce experimental difficulties. If the time between impacts is a significant portion of the capture duration, truncation problems are likely to arise. Furthermore, the oscillatory nature of the force spectrum can

potentially introduce severe instrumentation signal to noise problems. This problem can be particularly severe as the magnitude of the two impacts approach each other, producing nulls in the spectrum.

The double hit situation dramatically increases the number of ensemble averages required to produce high confidence spectral estimates. In an actual modal test as few as five impacts are ensemble averaged to estimate the spectra. This low number of ensemble averages is feasible as a result of the very repeatable nature of the input and output responses. In other words the input and output data can almost be considered deterministic, negating the need for frequency domain ensemble averaging. However, averaging is still utilized to minimize the effects of any extraneous noise within the instrumentation or system. When the double hit is encountered the repeatability characteristic is lost. Because of the many factors which control the creation of the second impact, its characteristics can often change significantly between subsequent impacts. Particularly variable is the delay between impacts, γ_0. Consider Fig. 17 which presents the force spectrum obtained with three different delays, $\gamma_0 = 0.04, 0.05, 0.06$ seconds, while holding all other parameters identical to those used in Fig. 16. The variation in each of the spectra with respect to one another is quite apparent. From a spectral estimation standpoint each of these curves represent one event which may be subsequently averaged to produce the force autospectrum. Because of the transform differences the variance of the spectral estimates also increases. Therefore to decrease the variance the number of averages must also must be increased. It is difficult to state exactly how many ensembles are necessary to produce low variance spectral estimates, however, at least an order of magnitude increase would be needed. This implies that to include double hit excitation in the ensemble, the total number of events is approaching that which is deemed acceptable for random excitation. Obviously the ease and simplicity associated with using only five or fewer events to obtain an accurate FRF is lost.

Typically experimentalists select impact excitation because of the relative ease in which impacts tests are performed. However when double hits are encountered, many of the attractive aspects of the method are lost: flat broadbanded excitation and low number of ensemble averages.
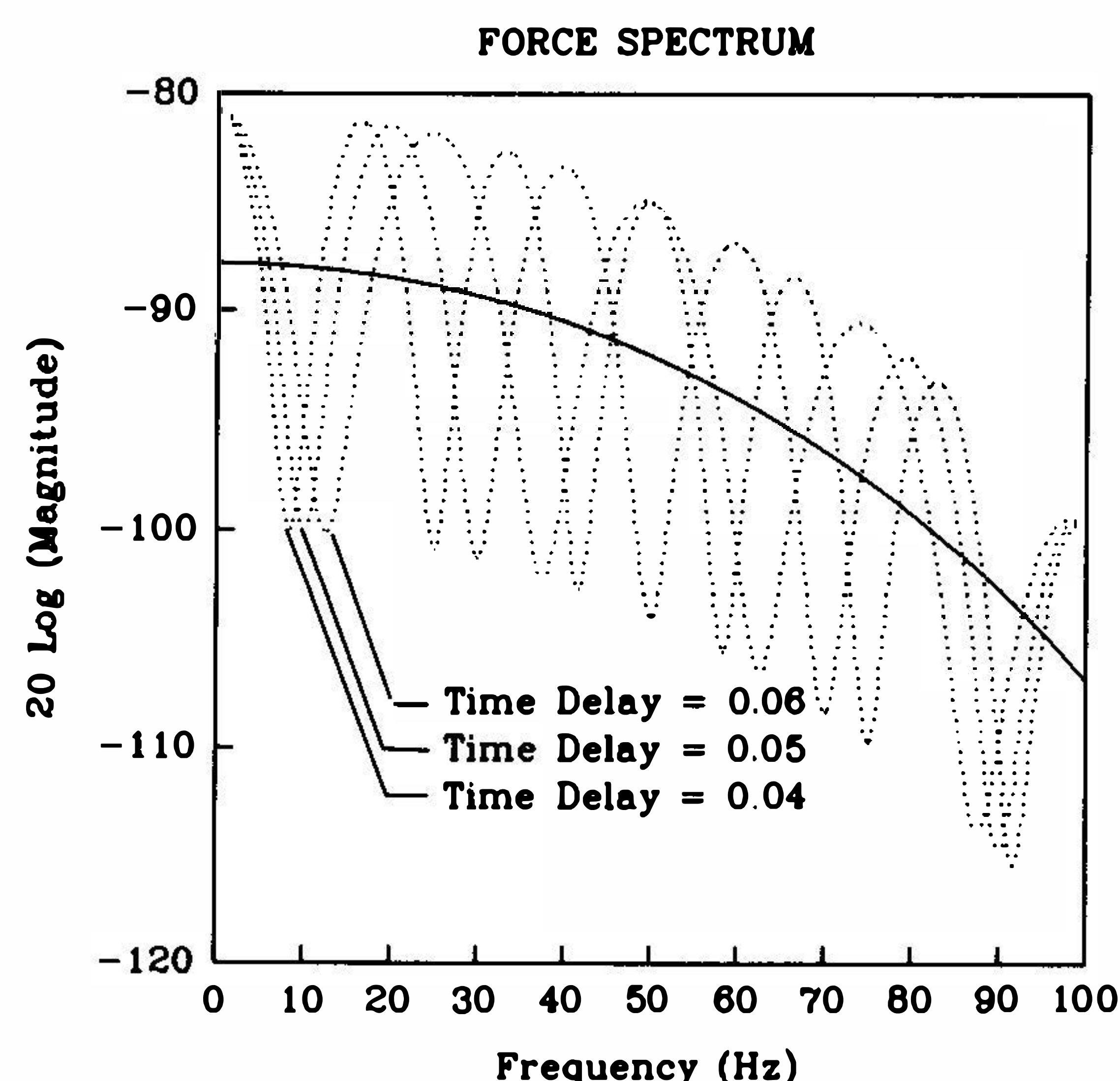

Fig. 17   Auto spectrum for a double hit half sine impact forces with T_1=0.01, F_1=1, F_2=0.5 (units of force) with three time delays=0.04, 0.05, 0.06 sec compared to the single hit spectrum of Fig. 3.

This paper has examined several aspects of FFT signal processing associated with hammer impact excitation. The method is quite robust, being sufficiently insensitive that accepted testing procedures usually keep any adverse effects to a minimum. If applied properly, the method is capable of exciting a broadband of frequencies with minimal equipment while providing high quality spectral estimates with only several ensemble averages. Obviously these aspects are what make the method so attractive and so widely used.

Accurate results can be obtained by impact tests provided the practitioner is cognizant of the hammer impact and vibration response characteristics in relation to the data acquisition specifications. The study has shown that the spectral data can be subjected to bias errors associated with an inadequate capture of the slowly decaying vibration signal. Analysis has indicated that provided at least six time constants vibration response data is acquired with a boxcar window the estimated FRF will be within 0.5% of the true value at a natural frequency. If the vibration decay is sufficiently long that this criteria can not be satisfied, an exponential window can be employed to artificially reduce any truncation effects. The exponential window, however, introduces another set of potential problems in terms of signal truncation and spectral leakage. The leakage is most detrimental because of the smearing effects between closely spaced modes. For the cases examined in this work, modal parameters estimated from FRF accentuated sufficiently that the leakage smearing and truncation characteristics were visible, proved to be very accurate. The accurate modal parameter estimation was attributed to the capabilities of the extraction algorithm.

In actual testing situations one cannot always rely on the extraction routine to compensate for deficiencies in the spectral data. Therefore, it is prudent to strive to obtain the highest quality spectral data possible. The results from this study can be used to establish practical time capture criteria to assure that the windowing/truncation bias error is minimized for impulse test data acquired under more realistic conditions. In general, the greater the duration of data captured the lower the error. For impulse response data acquired with a boxcar window, an accurate (within 0.5 % of the theoretical values over a baseband frequency range double the natural frequency) response function is estimated if 6 time constants of data is captured. It is advisable to increase the data capture duration rather than apply an exponential window to slowly decaying vibration response. Data acquired with an exponential window is colored by the process potentially making the modal parameter extraction more difficult. From a practical standpoint it may be difficult to back calculate the modal damping ratio because of numerical inaccuracies associated with the computations.

The examination of the double hit phenomena in impact testing demonstrated that their presence do not preclude the accurate determination of spectral functions. However, in order to do so potential signal to noise problems must be overcome and the number of ensemble averages used in the estimation process be increased dramatically. Therefore, in practice it is usually more convenient to simply reject a double hit from the ensemble.

In general, the entire impulse testing, data acquisition, and modal extraction process is rather forgiving and robust. By following accepted testing procedures the practitioner can usually obtain high quality spectral functions through impact testing.

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