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ABSTRACT

This paper presents the theory and an evaluation of a spatial sine testing parameter estimation algorithm that uses directly the measured forced mode of vibration and the measured force vector. The parameter estimation algorithm uses an ARMA model and a recursive QR algorithm is applied for data reduction. In this first evaluation, the algorithm has been applied to a frequency response matrix (which is a particular set of forced mode of vibration) using a sliding frequency window. The objective of the sliding frequency window is to execute the analysis simultaneously with the data acquisition. Since the pole values and the modal density are obtained from this analysis during the acquisition, the analysis information can be used to help determine the forcing vectors during the experimental data acquisition.

[redacted]

$[A]$ autoregressive coefficients
$[a]$ reduced autogressive coefficient
ARMA autoregressive moving average
$[B]$ moving average coefficients
$[b]$ reduced moving average coefficients
$[C]$ damping matrix
$[c]$ reduced damping matrix
$[F]$ forcing matrix
$[FV]$ reduced force matrix
$[f]$ forcing matrix
$[I]$ idenity matrix
j $\sqrt{-1}$
$[K]$ stiffness matrix
$[k]$ reduced stiffness matrix
$[L]$ modal participation matrix
$[M]$ mass matrix
$[m]$ reduced mass matrix
$[Q]$ the q matrix from the QR reduction
$[R]$ upper triangular matrix from the QR reduction
s Laplace domain variable
$[U]$ right singular vectors
$[V]$ left singular vectors
$[\ddot{x}(t)]$ forced mode of vibration acceleration matrix in time domain
$[\dot{x}(t)]$ forced mode of vibration velocity matrix
$[x(t)]$ forced mode of vibration displacement matrix
$[Y]$ eigenvectors in reduced space
$\{y\}$ eigenvector

Greek

α $n \times L$
$[\alpha]$ projection of $\overline{X}$ on the Q space
$[\beta]$ part of $[\overline{X}]$ laying outside the Q space

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λ pole value or eigenvalue
ν number of online estimated modes
$[\Sigma]$ diagonal matix of singular values
$[\Theta]$ reduced response matrix
$[\Lambda]$ diagonal matrix containing the system poles
$[\Psi]$ mode shape matrix
ω frequency

Sub - Superscripts

L number of spectral lines included in frequency window
N number of active normal modes in the window
N_f number of independent forces in the window
N_i number of inputs
N_o number of responses
n number of forced modes of vibration per spectral line
$[\ ]^H$ Hermitian
$[\ ]^T$ transpose

Other Symbols

$\overline{X}$ (overbar) forced mode of vibration to be added to the new analysis
$\underline{X}$ (underbar) forced mode of vibration to be deleted from the analysis
$\hat{X}$ forced mode of vibration to be kept in the analysis

1. Concepts of the Spatial Sine Testing

Historically in experimental vibration testing, sine testing is normally the synonym for normal mode or phase-resonance testing, which is a test procedure that involves single or multiple monophase input (typically eight or less). The method is based on exciting the structure at its natural frequency and modifying the phase and amplitude of the force vectors until the response is 90 degrees out of phase with the force. After the transients dissipate, the structure's response vector is in a pattern equal to a normal mode. This tuning procedure is then repeated at each natural frequency to obtain the associated normal modes of the system. The major advantage of this testing method is the measurements have a very good signal to noise ratio. Also, nonlinear aspects of each mode are amendable to investigation by sine testing. The disadvantage is the tedium involved in tuning the force vector to each of the natural frequencies. When it is not possible to measure all responses simultaneously, the process must be repeated for each set of responses. Therefore, normal mode sine testing, referred to as classical sine testing, has been traditionally very time consuming. With the introduction of digital technology and larger acquisitions systems, the ability to measure all points concurrently has become economically feasible. The advantages of sine testing coupled with the lower cost of implementation, has renewed interest in sine testing.

At the University of Cincinnati "Spatial Sine Testing" is being investigated [1,2]. The main difference between spatial sine testing and classical sine testing is the method in which the data is acquired. The main idea behind spatial sine testing is to excite the structure with an arbitrary force vector at a particular frequency and measure the force vector and resulting forced mode of vibration. The structure's response will be a linear combination of the normal modes in the vicinity of this frequency. The parameter estimation algorithm then extracts the underlying modal parameters from the measurements. Therefore, it is not necessary to tune the force vector to obtain a normal mode response, and an open loop force control system is appropriate. This is a major deviation from the classical sine testing in the number of intended input locations as well as in the absence of force control loops, which are sometimes necessary for classical sine testing .

There is also a difference in the acquisition procedure. In the classical sine testing, one moves from one natural frequency to another and excites the normal modes. For spatial sine testing, the data acquisition and the data analysis are separate tasks. While the acquisition of a new data set is in progress, the previously measured data sets can be analyzed. A subset of the measured data, which is determined by the frequency

window, is analyzed for the system properties, the pole values and the mode shapes. The frequency band of the analysis window can be changed based on the previous on-line analysis. When the acquisition moves up in frequency, so does the analysis. Therefore, for the analysis, there is a frequency window that spans a certain frequency range. This window is variable in length and shifts in frequency together with the acquisition. This windowing concept for spatial sine testing is new for sine testing, as well as the fact that the analysis occurs in parallel with the acquisition. The sliding window concept provides for redundant estimates of modal parameters so that stability and accuracy measures may be obtained. Therefore, this method does not take advantage of the normal mode excitation capabilities inherent to sine test excitation.

The parameter estimation algorithm is based on the estimation of the underlying normal modes in the force mode of vibration. This is only possible when the number of independent force vectors is equal to or exceeds the number of active modes. Since the analysis occurs simultaneously with the acquisition, a decision can be made if more or less force vectors are needed at a certain spectral line. The frequency step and force pattern for the acquisition can be determined based upon the modal density in the frequency window. In the case of high modal density, a small frequency increment and more force patterns should to be taken, while for low modal density, larger frequency steps and fewer force patterns can be taken. Therefore, the analysis can control the number of force vectors and applied forces, as well as the frequency steps. Spatial sine testing is in some ways similar to the "Multi Phase Sine Testing" [3,4] method, except that the number of exciters is normally much larger and there is no closed loop force control.

2. Theoretical Derivation of the Algorithm

The paramter estimation algorithm is based upon the standard formulation on Newton's second law [5]

$$[M][\ddot{x}]+[C][\dot{x}]+[K][x]=[f]$$

or in the Fourier Domain

$$-\omega^2[M][x]+j\omega[C][x]+[K][x]=[f] \tag{1}$$

Th known paramters in Eq. (1) are the force vector matrix and the forced mode of vibration matrix. The unknown parameters are the mass, damping and stiffness matrix. Once the mass, damping and stiffness matrices are known, the eigenvalues and eigenvectors of the system can be found by solving Eq. (2)

$$\left\{\lambda^2[M]+\lambda[C]+[K]\right\}\{y\}=\{0\} \tag{2}$$

where λ is a pole or eigenvalue of the system and $\{y\}$ is the eigenvector of the system associated with the pole value λ.

A problem with above equations, is that the mass, stiffness and damping matrices can become very large for many systems. The dimensions of these three matrices are directly related to the number of measurement points on the structure. As a consequence, the solution of Eq. (2) might be very time consuming and ill-conditioned. Another problem is that the measured forced mode of vibration matrix contains only information from one spectral line. It is probable that not all modes of the system will be active at this frequency. Since the measured forced mode of vibration matrix is a linear combination of the active normal modes, the estimated mass, damping and stiffness matrices will not contain information related to the inactive modes. In fact, it would be impossible to estimate a full rank mass, damping and stiffness matrix

from Eq. (1). This means that the dimensions of the mass, damping and stiffness matrices in Eq. (1) should not be larger than the number of active poles. Therefore, there must be a transformation possible such that the dimensions of the unknown matrices in Eq. (1) are reduced to the number of active poles at the considered frequency.

Equation (1) can be expanded for many spectral lines

$$\begin{cases} -\omega_1^2[M][x]_1 + j\omega_1[C][x]_1 + [K][x]_1 = [f]_1 \\ -\omega_2^2[M][x]_2 + j\omega_2[C][x]_2 + [K][x]_2 = [f]_2 \\ \vdots \\ -\omega_L^2[M][x]_L + j\omega_L[C][x]_L + [K][x]_L = [f]_L \end{cases} \tag{3}$$

or in matrix notation

$$[[M][C][K]]\begin{bmatrix} -\omega_1^2\left[\,I\,\right][x]_1 & -\omega_2^2\left[\,I\,\right][x]_2 & .. & -\omega_L^2\left[\,I\,\right][x]_L \\ j\omega_1\left[\,I\,\right][x]_1 & j\omega_2\left[\,I\,\right][x]_2 & .. & j\omega_L\left[\,I\,\right][x]_L \\ [x]_1 & [x]_2 & .. & [x]_L \end{bmatrix} = \begin{bmatrix} [f]_1 & [f]_2 & .. & [f]_L \end{bmatrix} \tag{4}$$

where $\left[\,I\,\right]$ is an N_0 by N_0 identity matrix and L is the number of spectral lines included in the frequency window.

Equation (5) results from taking the transpose of Eq. (4) and noting that $\left[-\omega^2\left[\,I\,\right]_{N_0 \times N_0}[x]\right]^T$ is equal to $-\omega^2\left[\,I\,\right]_{n\times n}[x]^T$.

$$\left[\left[\omega^2\right][X][j\omega][X][X]\right]\begin{bmatrix} [M]^T \\ [C]^T \\ [K]^T \end{bmatrix} = [F] \tag{5}$$

where

$$[X]_{\alpha \times N_0} = \begin{bmatrix} [x]_1^T \\ [x]_2^T \\ \vdots \\ [x]_L^T \end{bmatrix}$$

α = number of measured forced mode of vibrations per spectral line $\times$ number of spectral lines included in the calculations ($\alpha = n \times L$)

$$[\omega^2]_{\alpha\times\alpha} = \begin{bmatrix} -\omega_1^2\left[\begin{smallmatrix}\ddots\\ & I\\ & & \ddots\end{smallmatrix}\right] & [0] & .. & [0] \\ [0] & -\omega_2^2\left[\begin{smallmatrix}\ddots\\ & I\\ & & \ddots\end{smallmatrix}\right] & .. & [0] \\ . & . & .. & . \\ . & . & .. & . \\ . & . & .. & . \\ [0] & [0] & .. & -\omega_L^2\left[\begin{smallmatrix}\ddots\\ & I\\ & & \ddots\end{smallmatrix}\right] \end{bmatrix}$$

$$[j\omega]_{\alpha\times\alpha} = \begin{bmatrix} j\omega_1\left[\begin{smallmatrix}\ddots\\ & I\\ & & \ddots\end{smallmatrix}\right] & [0] & .. & [0] \\ [0] & j\omega_2\left[\begin{smallmatrix}\ddots\\ & I\\ & & \ddots\end{smallmatrix}\right] & .. & [0] \\ . & . & .. & . \\ . & . & .. & . \\ . & . & .. & . \\ [0] & [0] & .. & j\omega_L\left[\begin{smallmatrix}\ddots\\ & I\\ & & \ddots\end{smallmatrix}\right] \end{bmatrix}$$

$\left[\begin{smallmatrix}\ddots\\ & I\\ & & \ddots\end{smallmatrix}\right]_{n\times n}$ is a $n\times n$ identity matrix

$$[F]_{\alpha\times N_0} = \begin{bmatrix} [f]_1^2 \\ [f]_2^T \\ \vdots \\ [f]_L^T \end{bmatrix}$$

When the assumption of Maxwell reciprocity is valid, the mass, damping and stiffness matrices are symmetric and the transpose could theoretically be dropped. However, since measured data is used, it is more complete to not to assume that the matrices are symmetric.

Note that the forced mode of vibration matrix [*X*] contains different estimates of the forced modes of vibration at different spectral lines. These forced modes of vibration are a linear combination of the active normal modes in the frequency range. Therefore, the number of independent vectors in the forced mode of vibration matrix is equal to the number of active modes. So, when a singular value decomposition is performed on the matrix [*X*], we should obtain

$$[X]_{\alpha\times N_0} = [U]_{\alpha\times N}\left[\begin{smallmatrix}\ddots\\ & \Sigma\\ & & \ddots\end{smallmatrix}\right]_{N\times N}[V]^H_{N\times N_0} \tag{6}$$

where $[U]$ = the right singular vectors, $\left[\begin{smallmatrix}\ddots\\ & \Sigma\\ & & \ddots\end{smallmatrix}\right]$ = singular values, N = number of active modes, and $[V]$ = the left signular vectors.

Substituting Eq. (6) into Eq. (5) gives

$$\left[[\omega^2][U]\left[\begin{smallmatrix}\ddots\\ & \Sigma\\ & & \ddots\end{smallmatrix}\right][V]^H \quad [j\omega][U]\left[\begin{smallmatrix}\ddots\\ & \Sigma\\ & & \ddots\end{smallmatrix}\right][V]^H \quad [U]\left[\begin{smallmatrix}\ddots\\ & \Sigma\\ & & \ddots\end{smallmatrix}\right][V]^H\right]\begin{bmatrix}[M]^T\\ [C]^T\\ [K]^T\end{bmatrix} = [F] \tag{7}$$

Postmultiplying Eq. (7) with $[V]$ and taking $[V]^H$ out of the coefficient matrix, results in

$$\left[\left[\omega^2\right][U]\left[\Sigma\right] \; [j\omega][U]\left[\Sigma\right] \; [U]\left[\Sigma\right]\right]\begin{bmatrix}[V]^H[M]^T[V]\\ [V]^H[C]^T[V]\\ [V]^H[K]^T[V]\end{bmatrix}=[F][V]$$

or

$$\left[\left[\omega^2\right][\Theta][j\omega][\Theta][\Theta]\right]\begin{bmatrix}[m]^T\\ [c]^T\\ [k]^T\end{bmatrix}=[FV] \tag{8}$$

where $[m]_{N\times N}$ = the reduced mass matrix, $[c]_{N\times N}$ = the reduced damping matrix, $[k]_{N\times N}$ = the reduced stiffness matrix, $[\Theta]_{\alpha\times N}$ = the reduced response matrix, $[FV]_{\alpha\times N}$ = the reduced force matrix and N is the number of active normal modes.

Equation (8) represents α equations containing $3N^2$ number of unknown parameters. In order to have a unique solution, α has to be equal or larger than $3N^2$. This condition can be satisfied by taking sufficient forced mode of vibration estimates at each spectral line and/or by including more spectral lines in the calculations. However, N can increase when more spectral lines are included in the frequency window. For the case when only one spectral line is included, the rank of the coefficient matrix in Eq. (8) will be N instead of $3N$, which yields a non-unique solution. Therefore, there has to be at least two spectral lines included in the calculation. When Eq. (8) is overdetermined, a singular value decomposition matrix or a normal equation method can be used to solve for the unknown reduced mass, damping and stiffness matrices.

Once the reduced system matrices are found, they can be substituted into Eq. (2) to yield

$$\left\{\lambda^2[m]+\lambda[c]+[k]\right\}\{y\}=\{0\} \tag{9}$$

A common way to solve Eq. (9), is to rewrite the equation into an eigenvalue problem by formulating a companion matrix. If one is interested in solving for the undamped modes shapes and natural frequency, Eq. (9) can be solved omitting the reduced damping matrix.

Equation (5) works well with analytical data because the response and force data are free of errors and the number of modes of the system matches the number of modes in the data. However, for experimental data, a better formulation, of Eq. (5) is to use an algorithm based upon an ARMA model of the form [6,7]

$$\sum_{r=0}^{r=ar}[A_r][X]^T\left[\omega^r\right]=\sum_{i=0}^{i=ma}[B_i][F]^T\left[\omega^i\right] \tag{10}$$

In the case of spatial aliasing, which means that not enough points are measured on the structure to define all the modes uniquely, Eq. (1) in combination with any data reduction technique based upon the spatial information will fail to predict all the poles. This problem can be avoided by increasing the *ar* coefficient in Eq. (10) to a value greater than two [8]. Increasing the number of terms on the right hand side of Eq. (10), will compensate for residual effects [8].

As mentioned earlier, the $[X]$ matrix spans a limited space equal to the number of active modes in the frequency window. In order to solve Eq. (1), a transformation must to be applied such that the $[A_r]$ coefficients become of dimension N instead of N_o. Previously, a singular value decomposition was suggested for this

transformation. However the same results are obtained using a QR decomposition. The advantage of the QR decomposition is its speed and the feasibility of recursive implementation. Applying a QR decomposition to $[X]$ and $[F]$ results in

$$[X]^T_{N_0 \times \alpha} = [Q]_{N_0 \times N}[R]_{N \times \alpha}$$
$$[F]^T_{N_i \times \alpha} = \left[Q_f\right]_{N_i \times N_f}\left[R_f\right]_{N_f \times \alpha} \tag{11}$$

TABLE 1 BODY IN WHITE ANALYSIS RESULTS

P.T.D.* Results	Sliding Window (10)* ma=2	ma=0	No Sliding Window (10)* ma=2	ma=0	Sliding Window (20)* ma=2	ma=0
22.442	22.4399	22.4293	22.4419	22.4432	22.4411	22.4425
	-22.4430	22.4436	-22.4377	-22.2975	-22.4376	-22.4305
0.7940	0.7902	0.7436	0.7964	0.7663	0.8005	0.7702
	0.7949	0.7702	0.7911	0.7373	0.7965	0.7454
25.106	25.1081	25.1058	25.1080	25.1084	25.1039	25.0930
	-25.1111	-25.0958	-25.1054	-25.0865	-25.1056	-25.1087
0.5835	0.5871	0.5647	0.5873	0.6179	0.5597	0.5763
	0.5852	0.5525	0.5847	0.5967	0.5715	0.5864
31.446	31.4451	31.4451	31.4461	31.4471	31.4457	31.4420
	-31.4451	-31.4464	-31.4444	-31.4322	-31.4461	-31.4365
0.04453	0.4451	0.4485	0.4435	0.4366	0.4461	0.4568
	0.4452	0.4531	0.4460	0.4129	0.4494	0.4495
35.452	-	35.4491	35.4519	35.4726	35.4428	35.6629
	-	-35.4276	-35.4538	-35.4574	-35.4247	-35.6857
0.7642	-	0.8043	0.7638	1.1159	0.5705	0.3157
	-	0.7815	0.7853	0.8172	0.3335	0.4074
36.458	36.4577	36.4552	36.4577	36.4594	36.4577	36.4384
	-36.4528	-36.4551	-36.4528	-36.4416	-36.4624	-36.4672
0.4883	0.4929	0.4862	0.4929	0.4648	0.4970	0.4672
	0.4793	0.4850	0.4793	0.4425	0.4982	0.4836
38.251	38.2259	38.2411	38.2604	38.2422	38.2663	38.2379
	-38.2601	-38.2579	-38.2612	-38.2548	-38.2564	-38.2556
0.569	0.5725	0.5539	0.5705	0.5688	0.5822	0.5839
	0.5714	0.5633	0.5805	0.5663	0.5849	0.5960

*P.T.D.: Polyreference Time Domain Technique; (10): analysis based upon 10 spectral lines; (20) analysis based upon 20 spectral lines.

Substituting Eq. (9) into Eq. (10) and premultiplying by $[Q]^H$ results in

$$\sum_{r=0}^{r=ar}[a_r][R]\left[\dot{\omega}^r\right]=\sum_{i=0}^{i=ma}[b_i][R_f]\left[\dot{\omega}^i\right] \tag{12}$$

where

$$[a_r]_{N\times N}=[Q]^H_{N\times N_0}[A_r]_{N_0\times N_0}[Q]_{N_0\times N}$$

$$[b_i]_{N\times N_f}=[Q]^H_{N\times N_0}[B_i]_{N_0\times N_i}[Q_f]_{N_i\times N_f}$$

Equation (12) can be solved for the unknown coefficients $[a_r]$. The pole values of the active modes are obtained from the characteristic equation

$$\sum_{r=0}^{r=ar}[a_r]\left[\dot{\lambda}^r\right][Y]=[0] \tag{13}$$

where $[Y]$ is the eigenvectors in the reduced space.

The $[a_r]$ matrices are normally real matrices. However, in the current implementation of the program, the $[a_r]$ matrices are found by solving the null space of the known coefficient matrix. This method is faster than solving Eq. (13) through a normal equation. The drawback however is slightly complex $[a_r]$ matrices, which cause non perfect complex conjugate poles. This is illustrated in Table 1. The physical modes are calculated by transforming the eigenvectors from Eq. (13) back to the physical space.

$$[\Psi]_{N_0\times N}=[Q]_{N_0\times N}[Y]_{N\times N} \tag{14}$$

3. The Sliding Window Concept and Implementation

An obvious implementation is to apply the parameter estimation algorithm simultaneously with the sine testing data acquisition where the analysis is based on the most current data. Upon completion of the data acquisition of a spectral line, the analysis is repeated based upon the newest acquisition and previously taken acquisitions. To minimize the size of the equations, the first acquisitions are deleted for the latest acquisitions in the analysis. It is possible to execute a QR decomposition on the new forced mode of vibration matrix following the procedure described above. However, the forced mode of vibration matrix changes insignificantly from one analysis to another since the forced mode of vibration is dominated by the same dominant modes. Therefore the space spanned by the forced mode of vibration does not change rapidly from one frequency window to another and the QR decomposition of the previous window can be used to obtain the QR decomposition of the current window [9]. Assume that $[\overline{X}]$ is the matrix with the forced mode of vibration to be added to the new analysis. The projection of this matrix on the Q space of the previous window is obtained by $[Q]^H[\overline{X}]$. The part of $[\overline{X}]$ laying outside the Q space is given by

$$[\beta]=\left[\overline{X}\right]-[Q][Q]^H\left[\overline{X}\right] \tag{15}$$

In case $[\overline{X}]$ contains any new active modes, $[\beta]$ is a non zero matrix, and has a rank different from zero.

As a consequence, the Q space has to be extended in order to span the same space as the forced mode of vibration matrix $[X\overline{X}]$. The number of vectors to be added to the Q space is equal to the rank of $[\beta]$. The vectors can be obtained from the significant Q vectors of the QR decomposition of the $[\beta]$ vector.

$$[\beta] = [Q_\beta][R_\beta] \tag{16}$$

The new QR decomposition of the extended forced mode of vibration matrix becomes

$$[X\overline{X}] = [Q\, Q_\beta]\begin{bmatrix}[R] & [\alpha]\\ [0] & [R_\beta]\end{bmatrix} \tag{17}$$

where $[\alpha] = [Q]^H[\overline{X}]$. The columns of all the matrices in Eq. (17) can be interchanged in order to obtain an upper triangular $[R]$ matrix.

A similar procedure can be applied to reduce the columns of the $[X]$ matrix and obtain an updated QR decomposition. Assume the first columns of the $[X]$ matrix have to be deleted

$$\begin{aligned}[X] &= [\underline{X}\,\hat{X}]\\ [\underline{X}\,\hat{X}] &= [Q][\underline{R}\,\hat{R}]\end{aligned} \tag{18}$$

where $[\hat{X}]$ is the columns of X to be kept and $[\underline{X}]$ is the columns of X to be deleted. The first step is to move the columns that have to be removed to the end of the response matrix.

$$[\hat{X}\,\underline{X}] = [Q][\hat{R}\,\underline{R}] \tag{19}$$

The $[\hat{R}\,\underline{R}]$ matrix in Eq. (19) can be made upper triangular again by performing a QR decomposition without pivoting on $[\hat{R}\,\underline{R}]$.

$$[\hat{R}\,\underline{R}] = [\underline{Q}][R'] \tag{20}$$

Substituting Eq. (20) into Eq. (18)

$$\begin{aligned}[\hat{X}\,\underline{X}] &= [Q][\underline{Q}][R']\\ [\hat{X}\,\underline{X}] &= [Q'][R']\end{aligned} \tag{21}$$

The last columns of the $[X]$ matrix and $[R']$ matrix can now be deleted and the updated QR decomposition becomes

$$[X] = [\hat{X}] = [Q'][R] \tag{22}$$

where $[R]$ is equal to $[R']$ without the last columns.

4. The Force Determination

As mentioned, the parameter estimation is performed simultaneously with the data acquisition. At any time during the data acquisition, the system parameters already estimated are known and a prediction of system parameters at higher frequencies may be available. The application of a forcing vector which enhances or suppresses certain system properties based upon the known system properties can be advantageous for the parameter estimation. This enables the parameter estimation to predict a system pole earlier in the process of to find a stable estimate of the system poles faster.

Consider that a particular system property, for example the first mode of the system, has been estimated, found stable and stored away to disc. By applying a force vector which enhances this system property, the response of the system will be dominated by this system property. The parameter estimation algorithm will extract the already known information. On the other hand, if a force was applied which suppresses the influence of this known system property in the response vector, the parameter estimation algorithm would be able to detect new system properties faster. Therefore, the first condition posed upon the forcing vector is that it should not enhance system properties already estimated and stored to disc. The ideal force vector is one that suppresses this information.

Assume that v stable modes have been estimated and stored to disc. Furthermore, assume that a forcing vector $\{F\}$ orthogonal to these modes has been obtained, such that

$$\begin{bmatrix} \Psi_1^T \\ \Psi_2^T \\ \vdots \\ \Psi_v^T \end{bmatrix}_{v \times N_0} \{F\}_{N_0 \times 1} = \{0\}_{v \times 1} \tag{23}$$

where Ψ_i the i-th mode shape vector $(N_o \times 1)$.

The response of a structure as a function of the system properties, is defined as [10]

$$\{X\} = [\Psi]\left[s\begin{bmatrix} \backslash & \\ & I \\ & \backslash \end{bmatrix} - \begin{bmatrix} \backslash & \\ & \Lambda \\ & \backslash \end{bmatrix}\right]^{-1} [\Psi]^T \{F\} \tag{24}$$

where $[\Psi]_{No \times N}$ is the mode shape matrix, $\begin{bmatrix} \backslash & \\ & \Lambda \\ & \backslash \end{bmatrix}_{N \times N}$ is the diagonal matrix containing the system poles and $\begin{bmatrix} \backslash & \\ & I \\ & \backslash \end{bmatrix}_{N \times N}$ is the identity matrix. Substituting the force vector of Eq. (23) into Eq. (24) results in

$$\{x\} = \left[\Psi_{v+1}\Psi_{v+2}..\Psi_N\right]\left[s\begin{bmatrix} \backslash & \\ & I \\ & \backslash \end{bmatrix} - \begin{bmatrix} \backslash & \\ & \Lambda \\ & \backslash \end{bmatrix}\right]^{-1} \left[\Psi_{v+1}\Psi_{v+2}..\Psi_N\right]^T \{F\} \tag{25}$$

where $\begin{bmatrix} \backslash & \\ & \Lambda \\ & \backslash \end{bmatrix}$ contains the poles λ_i, $i = v + 1,.,.,.,N$ and $\begin{bmatrix} \backslash & \\ & I \\ & \backslash \end{bmatrix}_{N\text{-}v \times N\text{-}v}$ is the identity matrix.

By applying a force orthogonal to the v modes, it is possible to obtain a response which is independent of the considered v modes. However, Eqs. (23) through (25) are derived for the ideal case where the number of inputs is equal to the number of degrees of freedom of the system. In practical situations, the number of forcing vectors is significantly smaller than the number of degrees of freedom. The expression for the response as a function of the force [10] then becomes

$$\{X\} = [\Psi]\left[s\left[\begin{array}{c} \\ I \\ \\ \end{array}\right] - \left[\begin{array}{c} \\ \Lambda \\ \\ \end{array}\right]\right]^{-1} [L]^T \{F\} \tag{26}$$

where $[L]_{N \times N_i}$ is the modal participation matrix and N_i is the number of inputs.

In order to cancel out the influence of the first v modes in the response, a forcing vector should be taken orthogonal to the associated v modal participation vectors. However, the parameter estimation algorithm does not estimate the modal participation. Since the objective is to suppress the known mode shapes, the following approximation for the modal participation matrix is acceptable. When the number of inputs is equal to the number of responses, the modal participation matrix $[L]$ is equal to the mode shape matrix $[\Psi]$. When there are fewer inputs than responses, the modal participation matrix can be approximated by a matrix equal to the mode shape at the input points. Since the mode shapes of the structure are known, the approximated modal participation matrix can be obtained from the stored mode shapes. Once the modal participation matrix is determined, one can estimate a vector orthogonal to this matrix and apply it to the structure.

When the number of modes in the modal participation matrix, becomes larger than the number of inputs, it may be impossible to find a vector which is orthogonal to the modal participation matrix. In this case, the modal participation matrix becomes full rank, assuming there is no spatial aliasing when the modes are reduced to the input locations. This implies that the null space associated with the modal participation matrix is empty. Since the orthogonal vectors are laying in the null space, it is impossible in this case to determine an orthogonal vector. Thus, the force, vector can only be orthogonal to a number of modes smaller than the number of inputs. Since the contribution of the last estimated modes is the most significant, it would be appropriate to make the force vector orthogonal to the last N_i - 1 stored mode shapes. If $[L']$ represents the modal participation matrix for the last N_i - 1 modes, the orthogonal forcing vector can be found by solving for the null space of the $[L']$ matrix. Instead, a vector orthogonal to the total mode shape can be estimated and then reduced to the input locations. The advantage of this procedure would be that all of the stable modes could be included in the determination of the orthogonal vector. However, this procedure is more time consuming and the obtained forcing vector is inferior in suppressing the known mode shapes to the orthogonal forcing vector obtained from the null space of the reduced modal participation matrix.

In addition to the stored system properties, there are system property estimates available, which at this point are not stable. A forcing vector can be applied which would enhance these properties and provide a stable estimate more quickly. The same rationale as in the previous paragraph can be used to prove this case. The only difference is that now a forcing vector equal to the mode shape at the input points, is applied. Since the modes shapes are not orthogonal with respect to each other, but with respect to the mass matrix, this forcing vector does not ensure that the only active component in the response will be the system pole which the force is trying to enhance.

5. The Rank Determination of the *R* Matrix

One of the most difficult tasks in the parameter estimation algorithm is the determination of the rank of the $[x]$ matrix in

$$[X] = [Q][R] \tag{27}$$

The rank of the $[R]$ matrix is equal to the rank of the $[X]$ matrix and, due to the properties of the $[R]$ matrix, is easier to determine compared to the $[X]$ for this matrix. For a QR decomposition with pivoting, the

diagonal of the [R] matrix contains the maximum entry for each row. Each column entry j or $r_{j,k}$ of the [R] matrix stands for the contribution of the q_j vector to the total response vector x_k. In the case of analytical data, the rank of a matrix is well defined. All rows in the [R] matrix with an index higher than the rank of the matrix, are zero. Therefore, for analytical data, the rank of the matrix can be determined based upon the first zero diagonal element of the [R] matrix. For experimental data, where noise is inevitable, the cutoff for the rank determination is not as obvious. It is very uncommon for experimental data to obtain an [R] matrix which has zero rows or zero diagonal elements.

For experimental data, a decision has to be made based upon the amplitudes of the diagonal elements of the [R] matrix. In recent algorithms, singular value plots [11, 12] have been used in the determination of the rank of a matrix. The diagonal elements of [R] have similar properties as the singular values. As long as the q_i has a significant contribution to the response, its associated $r_{i,i}$ value is approximately of the same order of magnitude as the previous $r_{j,j}$ coefficients. When q_i compensates only for the noise in the data, its $r_{i,i}$ is several magnitudes smaller than the other contributing $r_{j,j}$ values for the other q_j vectors. Therefore, by tracking the changes in magnitude of the diagonal elements of the [R] matrix, it is possible to determine the rank of the [R] matrix and as a consequence the rank of [X].

For a QR decomposition without pivoting, one can not use the diagonal elements of the [R] matrix to determine the rank. Instead of the diagonal, the maximum entry of each row has to be considered in the determination of the rank of the matrix. A program has been implemented to determine the rank of the matrix automatically. It sorts the diagonal elements in decreasing order. If the ratio of the last diagonal element over the first diagonal element is larger than 10%, the matrix is assumed to be full rank. Practically this means, that the last q_i vector still contributes at least 10% to the response. This implicitly means that the noise levels on the response are assumed to be smaller than 10% of the main component in the response.

When the matrix is not full rank, two checks are performed to determine the rank of the matrix. The first check is the ratio between the diagonal value $r_{i,i}$ and $r_{1,1}$. If this ratio is larger than 1%, the $r_{i,i}$ diagonal value is accepted for the next check. If the diagonal value $r_{i+1,\,i+1}$ is non zero, the ratio is taken between $r_{i,i}$ and $r_{i+1,\,i+1}$. If this ratio is larger than the maximum of the ratio between previous diagonal values, the rank of the matrix is reset to i. If $r_{i+1,\,i+1}$ is zero, the rank is set to i and the program is terminated. These tests are repeated for all diagonal elements of the [R] matrix. When none of the diagonal elements is zero, the rank is set to the diagonal entry for which the ratio was a maximum.

This rank evaluation is relatively accurate. However, since this is the most crucial decision in the algorithm, the user is provided with the diagonal r values and the rank estimate. At that point, the user can confirm this rank estimate, or change the rank estimate to a more appropriate rank estimate, based upon evaluation of the diagonal r elements.

6. Results

To evaluate the algorithm, it has been applied to a known measured structure, a body in white. The measurements consist of frequency response functions instead of the forced mode of vibration matrix and forcing vectors. This provides the possibility of performing the parameter estimation with a more traditional method, such as the polyreference time domain technique, as well as with the proposed sine testing parameter estimation algorithm. For this algorithm, the forced mode of vibration matrix is taken equal to the frequency response function matrix, while the force vector is equal to the identity matrix.

The algorithm is applied in several ways on the data of the body in white. First an ARMA model of the order ar=2 and ma=0 (Eq. (12)) is applied for a sliding constant width frequency window of 10 spectral lines. Four inputs are applied to the structure. The frequency window is moved 5 spectral lines forward for each new analysis. Since the moving average part of Eq. (12) compensates for residual effects, an ARMA model ar=2 and ma=2 is applied on the same data and frequency window size. For both cases, the pole

estimates change from one frequency window to the next window. Both cases are able to predict modes inside and outside the frequency window. In general, a better estimate is obtained for the higher frequency outside modes than for the low frequency outside modes. The main difference between Case 1, with no residual compensation, and Case 2, with residual compensation, is that the damping estimate of the modes ahead of the frequency window converges faster toward the true value with every new analysis. In both cases, it can be noticed that once a frequency window is past the natural frequency of a pole, the estimate of this particular pole degrades quickly (especially the damping estimate). Case 2 tends to be more stable for the estimation of the lower outside poles. It also has fewer computational pole estimates when a window is centered around a natural frequency.

A QR decomposition is performed on the complete forced mode of vibration matrix of the first frequency window. After this initial QR decomposition, the QR updating technique is applied each time the sliding frequency window moves up in frequency. In order to investigate the difference between the updating technique and a complete QR decomposition on the forced mode of vibration matrix $[X]$, the algorithm is applied on a single, non-sliding, frequency window of 10 spectral lines around the natural frequency of each pole. This is done with an ARMA model ar=2 and ma=0 as well as ar=2 and ma=2. In order to see the influence of the frequency window size, Case 1 and Case 2 are repeated, using a frequency window of 20 spectral lines. The same conclusions can be drawn for the influence of residual terms as for Case 1 and Case 2 with 10 spectral lines. The main difference between the 10 versus 20 spectral line window, is the later has fewer computational poles and is able to predict the higher outside frequency poles more accurately. The results of the 6 different cases can be found in Table 1.

It can be noticed that the algorithm computes non-conjugate pairs. This is a main difference with respect to the algorithms such as polyreference time domain, Ibrahim time domain and polyreference frequency domain. The main reason for the non-conjugate poles is due to the fact that the ARMA coefficients in Eq. (12) are calculated based upon the null space of the known coefficient matrix. Except for the 35 Hz pole, it can be concluded that the natural frequency estimation is very stable and independent of the frequency window size or residual effects. As can be expected, the damping estimation is more case dependent. Including the residual terms (ma=2) tends to improve the damping estimate. The algorithm has problems determining the 35 Hz pole, a local mode which is weakly excited. The No Sliding Window case is able to estimate the 35 Hz mode accurately while both Sliding Window cases have problems in estimating this pole. This indicates that for local modes or not-well-excited modes, the recursive QR algorithm loses enough accuracy to poorly estimate them.

7. Conclusions

The initial results of the algorithm look promising. However, there are aspects that need some further investigation such as:

- The sensitivity to the rank estimate of the $[x]$ matrix. In the case where the rank is too highly overestimated (f.e. rank=2, rank estimate=15), the algorithm becomes unstable and does not find the poles in the frequency window. In the case where the order is underestimated, the algorithm estimates the poles in the window; however, they do not appear in complex conjugate pairs.
- The automatic determination of the order of the ARMA model.
- The determination of the pole estimate. When the window is sliding, the estimate of the poles changes. The problem is how to decide which pole estimate to retain.
- The influence of non-linearities. The presence of non-linearities in the data could cause it to be analyzed inconsistently and might make the pole estimation algorithm become unstable.

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