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Saddle Path Stability, Fluctuations, and Indeterminacy in Economic Growth

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Abstract. *We present a macroeconomic growth model in which investment in physical capital exhibits positive externalities which raise the stock of knowledge. Treating physical capital and knowledge as two separate variables, we show that the model can generate endogenous growth. It is demonstrated that there exist at most two balanced growth paths (BGPs) with endogenous growth. If the BGP is unique it is always saddle-point stable. If there are two BGPs, the first is always stable in the saddle point sense, whereas the second cannot be a saddle point. Instead, the second is either totally stable, giving rise to local indeterminacy, or completely unstable. Further, we can demonstrate that a Hopf bifurcation may occur at the second BGP, leading to persistent fluctuations. Besides local indeterminacy, the model may also give rise to global indeterminacy.*

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1 Introduction

Recently, there have emerged two different approaches to endogenous growth. One approach emphasizes the creation of human capital (Lucas 1988) or knowledge capital (Romer 1990) as the source of technical change and, thus, perpetual growth. In the other view, referring back to Arrow (1962a), as for example in Romer (1986b), technological knowledge and perpetual growth results from learning by doing and knowledge spillover.

In the first view, resources must be committed to the development of new knowledge. In the Lucas (1988) variant, perpetual increase in per-capita income is due to allocating time to education and the creation of human capital. In the other variant, the R&D model of technical change, as for example the models proposed by Romer (1990) and Grossman and Helpman (1991), R&D spending creates new technological knowledge, improving product quality or expanding product variety and generating persistent per-capita growth.

A problem with the latter approach is, however, as Young (1993) has pointed out, that an investment into new technologies might not necessarily develop its full productive potentials at the moment of the investment. On the contrary, at the time of the investment, when other competing technologies exist, new technologies may not necessarily be superior to old ones. In fact, as Young (1993) has shown, new technologies are often less efficient than existing technologies at the time of their invention and implementation.[1]

The learning-by-doing approach to technical change stresses the learning aspect when new investments with new technologies are undertaken. This approach states that it is the learning period of a new technology that develops its new productivity potentials. Learning means the accumulation of improvements and experience. There are at least three aspects of how the efficiency can increase. First, incremental improvements on the invested capital goods are made, leading to full efficiency of the new capital goods. Second, the learning of skills to operate the new capital goods corresponds to the incremental improvements (on-the-job experience and training). Third, the improved capital goods and the skills may generate spillover effects to the efficiency of older vintages of capital goods or other production activities of agents.

In general, we should, however, admit a learning pattern where the accumulation of knowledge occurs with a delay. A formal theory of learning with various delay patterns is presented in Sato and Suzuwa (1983). By employing this study, we will posit a learning pattern with a simple delay structure. Concerning such a view on learning and technical change—as initiated by Arrow (1962a) and generalized by Levahri (1966)—Romer (1986b), however, has demonstrated that per-capita variables may grow without an upper bound if the spillover effects are large. In our model, however, we show that learning effects may be bounded.

Different from the common one-sector growth model with learning (see, for example, Romer [1986b]), our study introduces a model of technical change where learning is complementary to investment but where the effect of investment on the building up of physical capital is separated from the effect on learning. In the model, then, with no investment there is no learning and accumulation of knowledge, and with no accumulation of knowledge there is no technical change in the long-run. The model converges toward a stationary economy with no increase in efficiency and per-capita output.[2] To simplify matters, we neglect the intentional allocation of resources for the creation of human capital, knowledge capital, or public capital, as is characteristic for the endogenous growth models of Lucas (1988), Romer (1990), or Barro (1990).

There are other important features of our model. Most of the literature on endogenous technical change has been concerned with an equilibrium analysis, and the dynamics have been neglected for a long period of time. Only recently have studies become available on the out-of-steady-state dynamics of endogenous growth models. As it turns out, the out-of-steady-state dynamics of the standard endogenous growth models are characterized by saddle-path stability.[3] On the other hand, it appears now as more certain that externalities generate multiple equilibria, indeterminacy, possibly local instability, and transitory, or even persistent, fluctuations. For standard one-sector models with external effects (nonconvexity of technology), it has been shown by Boldrin and Rustichini (1994) that such phenomena can arise only when negative externalities are assumed. Our model, exhibiting positive feedback effects, however, allows for the above phenomena.

We show the existence of global and local indeterminacy.[4] Global indeterminacy may arise in the case of multiple equilibria, implying that the initial level of consumption, which can be chosen freely, crucially determines the long-run economic paths that the output and efficiency will take. Therefore, two different economic situations characterized by the same starting values for the capital stocks may reveal completely different long-run paths and growth rates. In terms of two economies, this means that one economy will always lag behind the other and will never catch up.

We also demonstrate the possibly local indeterminacy of economic equilibria. In this case, two economies with identical initial conditions with respect to the capital stock exhibit in the limit the same growth rates, but the transitional dynamics—and thus the transitory growth rates—depend on the starting values of consumption, which may be chosen freely by the economic agents. This parallels a result obtained by Boldrin and Rustichini (1994) where, however, a discrete-time version of a model with externalities is explored.

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[1]A similar argument might be made with respect to the creation of human capital. To increase efficiency in production, on-the-job training and experience might be necessary for human capital to become effective.

[2]Empirically, such an approach can find evidence in a recent study on the relation between investment in equipment and per-capita growth; see DeLong and Summers (1991).

[3]See Caballe and Santos (1993) and Mulligan and Sala-i-Martin (1993) for the Lucas model and Asada, Semmler, and Novak (1995) for the Romer model, where the saddle-point property of those models is proved. There is also some literature that studies the transitional dynamics for extended versions of endogenous growth models; see for example Benhabib and Perli (1994) for the Lucas model and Benhabib, Perli, and Xie (1994) for the Romer model; for a survey, see also Flaschel, Franke, and Semmler (1996), chapter 5.

[4]These terms have been introduced by Benhabib and Perli (1994).

Moreover, in our case we may also observe transitory oscillations in the growth rate if the eigenvalues of the Jacobian at the steady state have imaginary parts.

Lastly, we can demonstrate that our model can also generate persistent fluctuations. Related results are obtained by Greiner and Hanusch (1994) for a conventional growth model, i.e., for a model with zero per-capita growth, and by Greiner and Semmler (1996) for a growing economy.[5]

The remainder of the paper is organized as follows. In Section 2, we present our model. Section 3 studies the dynamic behavior of our economy. It is shown that there may be two steady states and indeterminacy of equilibria with transitory oscillations of the endogenous variables. Moreover, necessary conditions for the emergence of stable limit cycles are derived. Section 4 presents numerical examples that demonstrate the analytical results, and Section 5 concludes the paper.

2 The Model

Our economy is represented by a household that maximizes its discounted stream of utilities arising from consumption, $C(t)$,

$$\max_{\{C(t)\}} \int_0^\infty e^{-(\rho-n)t} u(C(t))\, dt, \tag{2.1}$$

subject to the budget constraint[6]

$$\dot{K} = w + iK - C - (\delta + n)K \tag{2.2}$$

where $u(\cdot)$ is a strictly concave utility function, $u'(\cdot) > 0$, and $u''(\cdot) < 0$. ρ denotes the constant rate of time preference, K the stock of physical capital, which depreciates with the rate δ, and i and w give the rate of return to capital and the wage rate respectively. The labor supply is assumed to grow with the constant rate n, and $L(0)$ is normalized to 1 so that all variables denote per-capita quantities.

The level of output of our economy is determined by a representative firm exhibiting a production function of the form $Y_a(t) = (A(t)L(t))^\alpha K_a(t)^{1-\alpha}$, with $Y_a(t) = L(t)Y(t)$ aggregate output, $K_a(t) = L(t)K(t)$ aggregate capital stock, and $A(t)$ individual stock of knowledge as accumulated experience. $\alpha \in (0, 1)$ is the coefficient in the Cobb-Douglas function determining the labor share in the production of output $Y_a(t)$. All variables are functions of time. In per-capita terms, the production function can be written as $Y(t) = A(t)^\alpha K(t)^{1-\alpha}$. Note that our specification of the production function implies that knowledge is a nonexcludable, but it rivals public good just as in the Lucas (1988) model.[7]

Per-capita output $Y(t)$ may be either consumed or invested, thus increasing the stock of physical capital in our economy. The firm is assumed to behave competitively, which gives the wage rate as $w = \alpha A^\alpha K^{1-\alpha}$ and the marginal product of physical capital as $i = (1-\alpha)A^\alpha K^{-\alpha}$.

As to the stock of knowledge $A(t)$, we assume that it is formed according to the learning-by-doing approach initiated by Arrow (1962a). In contrast to Arrow, however, who uses a vintage approach with fixed coefficients, we assume in our model that technical change is disembodied, and the production function is not restricted to fixed coefficients (see Levhari 1966). Moreover, we suppose that the contribution of gross investment to the formation of knowledge further back in time is smaller than the recent gross investment. This assumption makes sense economically, and can be formalized by defining the stock of knowledge as an integral of past gross investment with exponentially declining weights put on investment flows further back in time (see Sato and Suzawa 1983; Ryder and Heal 1973; and Feichtinger and Sorger 1988).

More specifically, we assume that the increase of knowledge induced by investment in the initial year is highest, and that it gradually decreases as time passes. $A(t)$ then is given by

$$A(t) = \varphi \int_{-\infty}^{t} e^{\varphi(s-t)} I(s)\, ds.$$

[5]The existence of persistent cycles was conjectured by Benhabib, Perli, and Xie concerning the Romer (1990) model, but not proved.

[6]In what follows we will suppress the time argument if no ambiguity arises.

[7]We are indebted to the referee for pointing out to us the relation of our formulation to Lucas (1988).

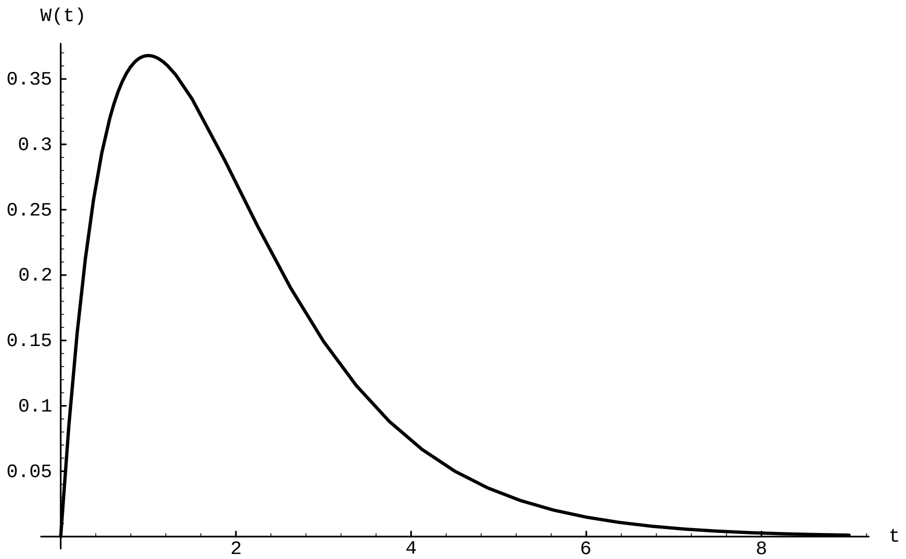

Figure 1
Weighting Function $W(t) = \varphi t e^{-\varphi t}$, with $\varphi = 1$.

The parameter φ represents the weight given to more-recent levels of gross investment. The higher φ, the larger is the contribution of more-recent gross investment to the stock of knowledge in comparison to flows of investment dating further back in time.

The weighting function we use here, however, is just one possibility among others. A more general function would be a gamma distribution function which is of the form $W(t) = \varphi t e^{-\varphi t}$. In this case, the weight put on investment flows—concerning its contribution to the actual stock of knowledge—first rises as we go back in time, but declines monotonously from a certain point in time. Or, stated another way, the effect of investment at $t = 0$ on the building up of knowledge first increases as time passes until it reaches a maximum, when it then declines. Figure 1 illustrates this fact for the gamma distribution function $W(t)$, with $\varphi = 1$. For a more detailed discussion as to the use of weighting functions, see Sato and Suzawa (1983), chapter 6.

It is worth noting that our formulation of learning by doing is closely related to those by Romer (1986b) and Sheshinski (1967). If population is constant and neither physical capital nor knowledge depreciate, we have $A = \varphi K$ for $A(-\infty) = K(-\infty) = 0$ and, thus, $Y = \varphi^{\alpha} K$, which is a special case of the Romer model.

Before using necessary conditions to describe the solution to this optimization problem, we first state that it can be shown that a solution to the household's problem exists if the rate of growth is bounded by a constant that is smaller than $\rho - n$. The proof is obtained by applying the theorem presented in Romer (1986a), and is available on request.

To describe the optimal solution, we can use Pontryagin's maximum principle. The current-value Hamiltonian for our problem is written as $H(\cdot) = u(C) + \mu(iK + w - C - (\delta + n)K)$.

Maximizing with respect to C yields $u'(C) = \mu$ for interior solutions. The evolution of μ is given by $\dot{\mu} = \mu(\rho + \delta) - \mu i$. Since the Hamiltonian is concave in its variables jointly, the necessary conditions are also sufficient if, in addition, the transversality condition at infinity

$$\lim_{t \to \infty} e^{-(\rho - n)t} \mu(t)(K(t) - K^{\star}(t)) \geq 0 \tag{2.3}$$

is fulfilled with $K^{\star}(t)$ denoting the optimal value.

As is usual in this sort of growth model with positive external effects, the solution to this optimization problem does not yield the socially optimal outcome. The latter would be achieved by explicitly taking into account an additional differential equation, giving the evolution of the stock of knowledge over time. The question, however, as to what measures should be taken to achieve the socially optimum solution, is beyond the scope of this paper.[8]

3 The Dynamics

The differential equation system for our economy is obtained from the necessary optimality conditions for the household, together with the equilibrium conditions determining the marginal product of physical capital as well as the wage rate. Further, the evolution of knowledge is obtained by differentiating $A(t)$ with respect to time. Thus, we get a three-dimensional differential equation system, which is given by

$$\dot{C} = A^{\alpha}K^{-\alpha}C\left(\frac{1-\alpha}{\sigma}\right) - C\left(\frac{\rho+\delta}{\sigma}\right) \tag{3.1}$$

$$\dot{K} = A^{\alpha}K^{1-\alpha} - C - (\delta+n)K \tag{3.2}$$

$$\dot{A} = \varphi A^{\alpha}K^{1-\alpha} - \varphi C - \varphi A, \tag{3.3}$$

with $-\sigma \equiv u''(C)C/u'(C)$ the elasticity of marginal utility, which is assumed to be constant.

It is obvious that sustained per-capita growth is only feasible if the external effect of investment is strong enough. In fact, if A is constant, we have the usual neoclassical growth model, with zero per-capita growth in the long run. This is well known, and has been pointed out by many authors before (see e.g., Romer 1986b or Sala-i-Martin 1990). Unfortunately, we cannot give general conditions that guarantee per-capita growth for our analytical model. This is one reason why we present some numerical examples below, where we show that this model is indeed capable of generating endogenous growth.

To explicitly investigate the steady-state and dynamic behaviors of our economy, we perform a change of variables with $k = K/A$ and $c = C/A$. Differentiating with respect to time gives $\dot{k}/k = \dot{K}/K - \dot{A}/A$ and $\dot{c}/c = \dot{C}/C - \dot{A}/A$. Our new system of differential equations in k and c is then given by

$$\dot{k} = k\varphi(1+c) + k^{1-\alpha} - \varphi k^{2-\alpha} - k(\delta+n) - c, \tag{3.4}$$

$$\dot{c} = c\varphi(1+c) + \frac{1-\alpha}{\sigma}k^{-\alpha}c - \frac{\rho+\delta}{\sigma}c - \varphi c k^{1-\alpha}. \tag{3.5}$$

A rest point of system (3.4)–(3.5) corresponds to a BGP of (3.1)–(3.3) with $\dot{A}/A = \dot{K}/K = \dot{C}/C = const.$ Let us, in a next step, examine whether system (3.4)–(3.5) has a steady state. It is immediately seen that $k = 0$ cannot be a steady-state value, since k is raised to a negative power in (3.5). This implies that there is no steady state with a zero value for k. Moreover, setting $c = 0$ and k so that $\varphi - (\delta+n) = k^{-\alpha}(\varphi k - 1)$ would yield a stationary point for (3.4)–(3.5). This, however, would imply that consumption is zero, a fact which does not make sense from the economic point of view, so we can exclude this rest point *a priori*, too. Therefore, we can consider the system (3.4)–(3.5) in the rates of growth and find its interior stationary points.

Let us, however, first demonstrate that for a specific situation, no balanced growth path (BGP) with sustained per-capita growth, i.e., a path on which all variables grow at the same constant rate, is feasible in our model. The following lemma gives the exact result.

Lemma. *If $\delta + n = \varphi$, no BGP with sustained positive per-capita growth exists.*

Proof. This lemma is proved as follows. First, we set $\delta + n = \varphi$ and compute c on the BGP from (3.4) with $\dot{k}/k = 0$. Inserting this c in $\dot{c}/c = 0$ gives an expression for $k^{-\alpha}$ on the BGP. Doing so, we get $k^{-\alpha} = (\sigma/(1-\alpha)) \cdot (-(\delta+n) + (\rho+\delta)/\sigma)$. Note that $k^{-\alpha} = A^{\alpha}K^{-\alpha}$. Then, inserting this expression in $\dot{C}/C$, derived from (3.1), yields the balanced growth rate as $\dot{C}/C = -\delta - n$. ■

Subsequently we exclude this possibility and suppose that the parameters are such that endogenous growth is possible. Proposition 1 shows that there may exist two BGPs or a unique BGP if this economy generates sustained per-capita growth with an endogenously determined growth rate.

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[8] For a detailed discussion on measures to be taken to improve the socially suboptimal solution of the decentralized problem so that it approaches the socially optimal solution, see Barro and Sala-i-Martin (1995), chapter 4.3.

Theorem 1. *(i) If $(\rho + \delta)/\sigma \leq \delta + n$, there exists a unique BGP in case of sustained per-capita growth. (ii) If $(\rho + \delta)/\sigma > \delta + n$, there exist two BGPs in case of sustained per-capita growth.*

Proof. see Appendix.

Proposition 1 tells us that we may have a situation with two BGPs. In that case, we can possibly observe global indeterminacy in the sense that the initial condition concerning consumption may crucially determine to which BGP the economy converges in the long run, given a fixed stock of physical capital and knowledge. That phenomenon arises if, for example, one BGP has one positive and one negative eigenvalue, while the other has only negative real parts of the eigenvalues. But the long-run growth rate in an economy also crucially depends on the initial conditions of k. Here we can speak of lock-in effects, in the sense of Arthur (1988) implying that an economy with a lower initial stock of knowledge possibly always lags behind other ones and can never catch up. A similar finding has also been reported in a paper by Futagami and Mino (1993) and an earlier paper by Shell (1967). These authors, however, could derive their results only for conventional growth models, i.e., for models with a zero per-capita growth rate.

We also can show that the above model can imply local indeterminacy of equilibrium paths. If the stationary state is (locally) completely stable, that is, all trajectories satisfying (3.4) and (3.5) which start in the neighborhood of this stationary state converge to the rest point, then there exists a continuum of paths $\{k(t), c(t)\}$ all converging to the stationary point. This holds because only the initial condition $k(0)$ is given for an economy, whereas the amount of initial consumption $c(0)$ can be chosen freely. Therefore, there exists a continuum of $c(0)$ satisfying the first-order conditions that are all feasible for the economy, so that we may say that the equilibrium path is indeterminate. What the level of $c(0)$ is that finally will be selected depends on noneconomic factors like cultural or institutional ones. They affect the transitional paths of the economy until it reaches the long-run balanced growth rate. Thus, the levels of long-run per-capita capital stock and consumption are also determined by $c(0)$, but of course, the long-run growth rate is not.

We may also observe transitory fluctuations of the growth rate if the eigenvalues of the Jacobian at the steady state are imaginary. Only in the long run, when we have convergence to the steady state, does the economy display growth rates that remain constant over time. On the other hand, an interesting question pertains to the persistence of fluctuating growth rates, and whether the economy reaches the steady-state growth rate at all.

To investigate the local dynamics, we proceed as usual and first calculate the Jacobian matrix. Using the fact that $\dot{k} = \dot{c} = 0$ at a stationary state, the Jacobian at the steady state is given by

$$J = \begin{bmatrix} c/k - \alpha k^{-\alpha} - k^{1-\alpha}(1-\alpha)\varphi & k\varphi - 1 \\ ((1-\alpha)c(-\alpha)k^{-\alpha-1}/\sigma) - c\varphi(1-\alpha)k^{-\alpha} & c\varphi \end{bmatrix}.$$

The eigenvalues of this matrix determine the local stability properties. Proposition 2 gives a complete characterization of the dynamics in our model.

Theorem 2. *(i) If there exists a unique BGP with endogenous growth, this path is always a saddle point. (ii) If there exist two BGPs with endogenous growth, the path associated with k_1^∞ is a saddle path, while the path associated with k_2^∞ can be anything except a saddle point. The points k_1^∞ and k_2^∞ denote the values of k on the BGP, and $k_1^\infty < k_2^\infty$.*

Proof. see Appendix.

This proposition characterizes both the global and local dynamics of our economy. It demonstrates that in the case of two BGPs, the first is always a saddle point, whereas the second may be either completely stable or unstable. Moreover, we can possibly also observe a stable limit cycle around the second BGP, which is the most complex dynamic behavior we can expect in our system. To show the existence of a persistent cycle, the Poincaré-Bendixson theorem could be applied where we need to demonstrate the existence of an unstable steady state and to find a compact invariant set that represents a trapping set for all trajectories. Here we prefer to resort to bifurcation theory and simulation studies to investigate the existence of persistent fluctuations.

The stability of the second BGP is determined by the sign of the trace of the Jacobian, $tr\,J$, and the determinant, $det\,J$. If $tr\,J < 0$ and $det\,J > 0$, the real parts of the eigenvalues are negative, indicating that the steady state is completely stable. Then we have a continuum of equilibria and may speak of local indeterminacy. It should be noted that this case can also provide an example for global indeterminacy, which arises if $k(0)$ is such that the choice of the initial value for $c(0)$ crucially determines to which BGP the economy converges in the long run. Moreover, if $(tr\,J)^2 - 4\,det\,J < 0$, the economy has imaginary eigenvalues, implying that it shows transitory oscillations in k (and of course c). This means that the endogenous growth rate also shows fluctuations until it reaches the steady state in the long run. To illustrate our analytical results, we will present a numerical example in the next section.

Of course, the second BGP can also be completely unstable, i.e., have two eigenvalues with positive real parts. In that case the first BGP is the only feasible solution. Then, the economy is both locally and globally determinate.

Before we present the numerical example, we note that our dynamic system may undergo a Hopf bifurcation, leading to stable limit cycles if we vary a certain parameter. If $trJ = 0$ and $det\,J > 0$, the system has two purely imaginary eigenvalues, and system dynamics may bifurcate into limit cycles. The economy then no longer converges to a balanced growth path, but instead shows persistent cyclical oscillations in the growth rate. To find the necessary condition for a Hopf bifurcation, we set $trJ = 0$. This yields $c^{\infty} = (\alpha k^{-\alpha} + (1-\alpha)\varphi k^{1-\alpha})/(\varphi + k^{-1})$. Substituting this c in $det\,J$ and knowing that $det\,J < 0$ must hold, we have the necessary condition for persistent cycles, $\alpha - u''(\cdot)C/u'(\cdot) < 1$.

For our analytical model, we check the necessary condition leading to a Hopf bifurcation. We leave aside the analytical study of whether sufficient conditions causing stable limit cycles are also fulfilled.[9] These are positive, crossing velocity of the eigenvalues and the sign of the coefficient determining the stability of the cycles.[10] We present some numerical examples in Section 4.

4 A Simulation Study

We illustrate our analytical findings by employing numerical examples.

4.1 Example 1

Let the instantaneous utility function be a function with a constant elasticity of marginal utility σ, with $\sigma = 0.4$. The coefficient in the Cobb-Douglas production function is set to $\alpha = 0.5$, and the depreciation rate $\delta = 0.19$. The population growth is assumed to be $n = 0.02$, and we set $\varphi = 1.65$. Interpreting one time period as two years then means that, for our value of φ, the contribution of investment two years back to the present stock of knowledge is $e^{-0.825 \cdot 2}$ or 19.2 percent, and the contribution of investment 5 years back is 1.62 percent.

The discount rate ρ serves as the bifurcation parameter. We analyze the range $0.0075 \le \rho \le 0.0412$. For these values of ρ there exist two steady states.[11] For $\rho > 0.0412$ there does not exist a stationary point of the dynamic system under consideration. As is well known, the eigenvalues of the Jacobian at the steady state are given by $\lambda_{1,2} = 0.5(tr\,J \pm \sqrt{D})$, with $tr\,J$ the trace of the Jacobian and $D \equiv (tr\,J)^2 - 4\,det\,J$.

It turns out that one of the steady states is always stable in the saddle point sense, as predicted by part (ii) of Proposition 2. For the second steady state, Figure 2 gives the trace of the Jacobian, trJ (upper curve) and D (lower curve) as a function of ρ.[12]

For about $\rho < 0.014$, both the trace as well as D are positive. Since $\sqrt{D}$ is smaller than $tr\,J$, the steady state is an unstable node. For $\rho \ge 0.014$, D becomes negative, but the trace of the Jacobian is still positive, implying that the steady state is an unstable focus. For $\rho_{crit} = 0.040899$, the trace is equal to zero. Since $D < 0$, this means that there are two imaginary eigenvalues with zero real parts. The steady state for this value of ρ is given by $k_1^{\infty} = 4.17066$ and $c_1^{\infty} = 1.02111$. Increasing ρ further, the trace of the Jacobian becomes negative, too. Since $D < 0$, this implies that the steady state now is a stable focus, until it vanishes for $\rho > 0.0412$.

For the value of $\rho_{crit} = 0.040899$, the dynamic system undergoes a Hopf bifurcation, and for a value slightly smaller than ρ_{crit}, stable limit cycles can be observed. Figure 3 shows the time path for $c(t)$ for a certain period of time, with ρ set to $\rho = 0.04075$.

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[9]Futagami and Mino (1995) present an endogenous growth model with public capital for which they also demonstrate the possibility of persistent cycles. These authors, however, only check whether two purely imaginary eigenvalues may occur.

[10]For details, see Guckenheimer and Holmes (1983). Below we compute those conditions numerically.

[11]For the numerical calculations of this part we used the software Mathematica (see Wolfram Research 1991).

[12]Note that rho stands for ρ in Figure 2.

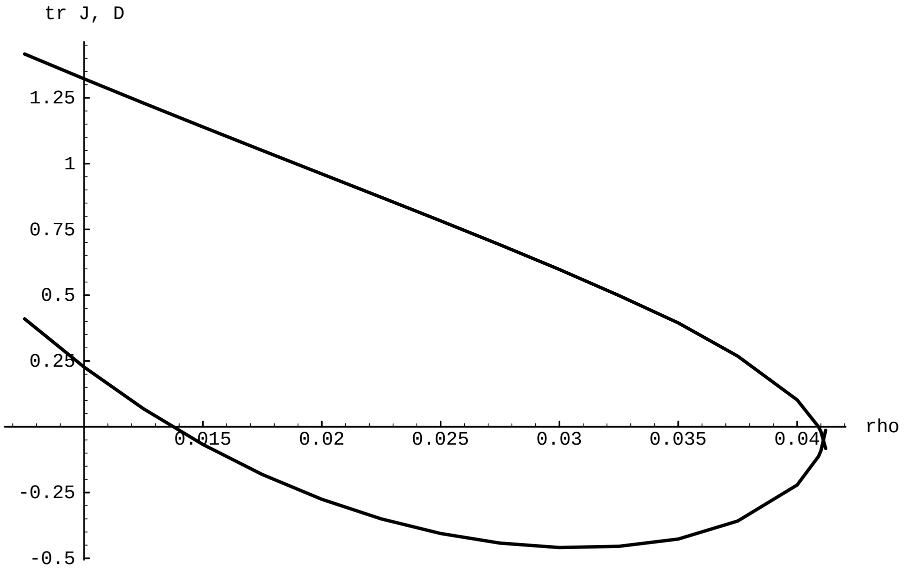

Figure 2
Trace of *tr J* and $(tr\,J)^2 - 4\,det\,J$ as a Function of ρ.

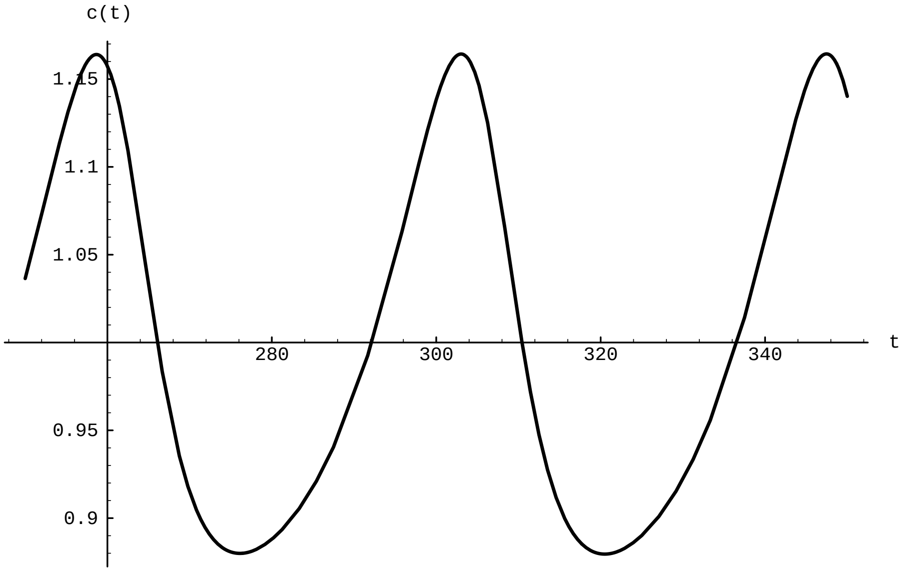

Figure 3
Time Path of $c(t)$ for a Certain Period of Time.

4.2 Example 2

We now present an example with different parameter values, and employ $\alpha = 0.3$. The coefficient in the utility function is now $\sigma = 0.5$. The depreciation rate is $\delta = 0.12$, the population is now assumed to be constant ($n = 0$), and φ is as above. Again, ρ serves as the bifurcation parameter.

Analyzing this system with $\rho = 0.2667$, we find two interior steady states. The first is given by $c_1^\infty = 1.78019$ and $k_1^\infty = 4.47098$, with eigenvalues $\lambda_1 = -0.248538$ and $\lambda_2 = 0.0975281$, indicating that this equilibrium is stable in the saddle point sense.

The second stationary point is $c_2^\infty = 1.93537$, $k_2^\infty = 4.79655$. The eigenvalues associated with this stationary point are $\lambda_{1/2} = -0.025915 \pm 0.157396\sqrt{-1}$, showing that this point is a stable focus.

If the economy converges to $\{c_2^\infty,\ k_2^\infty\}$, it will show transitory oscillations until it reaches the stationary value. Further, if we vary the discount rate ρ, we see that for $\rho = \rho_{crit} = 0.266452$, the real parts of the

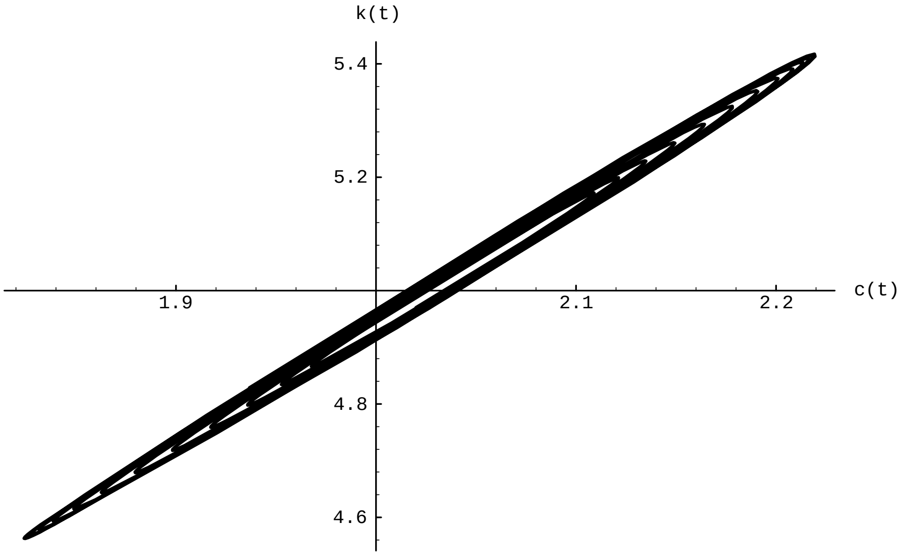

Figure 4
Limit Cycle in the $c(t) - k(t)$ Phase Diagram.

eigenvalues are zero. The stationary point for this value of ρ is shifted to $k^{\infty} = 4.97537$ and $c^{\infty} = 2.01863$. Since $\partial Re\lambda(\rho)/\partial\rho < 0$ for $\rho = \rho_{crit}$, the crossing velocity is nonzero, indicating a Hopf bifurcation. For this value of the discount rate, the other rest point is given by $c_1^{\infty} = 1.70687$ and $k_1^{\infty} = 4.32076$, with the eigenvalues $\lambda_1 = -0.339087$ and $\lambda_2 = 0.139892$, indicating that this equilibrium is still stable in the saddle point sense.

For $\rho = 0.266348$, we could again observe stable limit cycles around the stationary point which changes to $k_2^{\infty} = 5.02978$ and $c_2^{\infty} = 2.04371$. Figure 4 shows the limit cycle in the $c(t) - k(t)$ phase diagram.

4.3 Example 3

Next we fix the discount rate at $\rho = 0.628$, and take φ as the bifurcation parameter. The rest of the parameters are set to $\alpha = 0.1$, $\sigma = 0.1$, $b = 1$, $n = 0$, and $\delta = 0.1$.[13] The critical value for φ at which the real part of the eigenvalues vanishes is given by $\varphi_{crit} = 1.665909$. The stationary point associated with this value is $k^{\infty} = 8.043804$ and $c^{\infty} = 5.514256$. Since $\partial Re\lambda(\varphi)/\partial\varphi = -0.1708215$, it is assured that the eigenvalues cross the imaginary axis. The program used can also calculate the coefficient determining the stability of the cycle, β_2, as well as the coefficient giving the direction of the bifurcation, μ_2. Since $\beta_2 = -15.35438 < 0$, the cycle is stable, and because of $\mu_2 = -44.94277 < 0$, the periodic solutions occur for $\varphi < \varphi_{crit}$. In Figure 5 we show how the trajectory approaches the limit cycle with $\varphi = 1.66$ and starting values $k(0) = 7.82$ and $c(0) = 5.49$.

It should be mentioned that part (ii) of Proposition 2 is again confirmed, because this system also has a second stationary point which is a saddle for all values of φ that we considered.

5 Conclusion

This paper showed that inventive investment and learning by doing are interdependent. Without investment, learning by doing effects on efficiency are bounded, and without learning by doing effects, there is no increase of efficiency in the long run and the economy converges toward a stationary state in terms of per-capita income.[14] Different delay patterns for the learning mechanism can be specified that are likely to generate different effects on the transitional and long-run increase in efficiency. By assuming a simple

[13] To analyze this system we resorted to the computer program BIFDD, which is decscribed in Hassard, Kazarinoff, and Wan (1981). This program could not successfully be applied to the first two examples.

[14] Note, however, as mentioned in the introduction, that we have purposely neglected other factors for growth such as the intentional allocation of resources for the creation of human capital, R&D, or public capital.

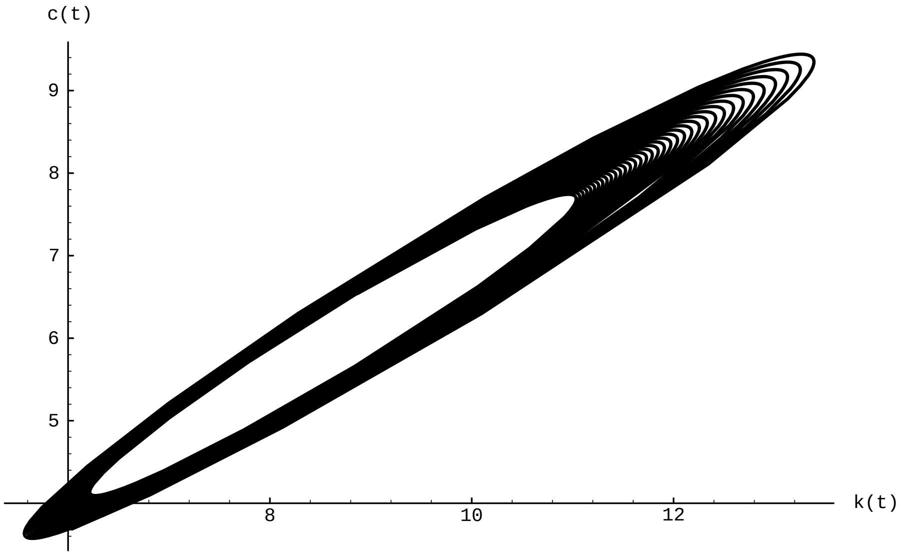

Figure 5
Limit Cycle in the $k(t) - c(t)$ Phase Diagram.

learning pattern where the strongest learning effects are complementary to the most recent investments, we obtain analytical results, confirmed by simulations, pertaining to the persistence of efficiency increase and per-capita growth. Moreover, we have seen that—and this is now well recognized by other studies on endogenous technical change—a model of endogenous technical change through externalities may admit rich dynamics such as local and global indeterminacy and persistent fluctuations. Such phenomena, as has been shown in the literature, cannot arise in standard one-sector growth models with positive externalities.

We want to note, however, that in the context of the present model we do not explore fluctuations at business-cycle frequency. Our growth model, although admitting differentials of growth rates across countries as well as fluctuating growth rates, is a model for the long run with all markets cleared instantaneously, and is thus not well suited to study short- and medium-run fluctuations. For the latter purpose, a nonmarket clearing approach with gradual adjustments appears to be better equipped.[15]

Appendix

Proof of Proposition 1

To prove Proposition 1 we compute c on the BGP from $\dot{k}/k = 0$ as $c^{\infty} = (k^{1-\alpha} - \varphi k^{2-\alpha} - k(\delta + n) + \varphi k)/(1 - \varphi k)$. Substituting c^{∞} in $\dot{c}/c$ leads to

$$f(k, \cdot) = k^{\alpha}\left(\varphi - \frac{\rho + \delta}{\sigma}\right) + k^{1+\alpha}\varphi\left(\frac{\rho + \delta}{\sigma} - (\delta + n)\right) - \left(\frac{1 - \alpha}{\sigma}\right)(\varphi k - 1).$$

A point for which $f(k, \cdot) = 0$ holds gives a BGP for our model.

For $k = 0$ we have $f(0, \cdot) = (1 - \alpha)/\sigma > 0$. Differentiating $f(k, \cdot)$ with respect to k gives

$$\frac{\partial f(k, \cdot)}{\partial k} = \alpha k^{\alpha - 1}\left(\varphi - \frac{\rho + \delta}{\sigma}\right) + (\alpha + 1)k^{\alpha}\varphi\left(\frac{\rho + \delta}{\sigma} - (\delta + n)\right) - \left(\frac{1 - \alpha}{\sigma}\right)\varphi.$$

To prove part (i) we note that $(\delta + \rho)/\sigma \le (\delta + n)$, and $\varphi \le (\delta + \rho)/\sigma$ implies $\partial f(k, \cdot)/\partial k < 0$ everywhere and $\lim_{k\to\infty} f(k, \cdot) = -\infty$ such that for this case, (i) is immediately seen.

If $(\delta + \rho)/\sigma \le (\delta + n)$ but $\varphi > (\delta + \rho)/\sigma$, part (i) is shown as follows. For $k \to \infty$ we have $\lim_{k\to\infty} f(k, \cdot) = -\infty$. Thus, $\partial f(k, \cdot)/\partial k < 0$ must hold at least locally. Since $\partial^2 f(k, \cdot)/\partial k^2 < 0$ holds

[15]See, for example, Flaschel, Franke, and Semmler (1996).

everywhere and is independent of k, $\partial f(k,\cdot)/\partial k > 0$ is not feasible once $\partial f(k,\cdot)/\partial k$ has become negative and, consequently, there is no second BGP.

If $(\delta+\rho)/\sigma > (\delta+n)$ and $\varphi \leq (\delta+\rho)/\sigma$, part (ii) is proved as follows: At the first BGP, the curve $f(k,\cdot)$ must cross the horizontal axis from above, i.e., $\partial f(k,\cdot)/\partial k < 0$ must hold at this point. For the second BGP, the curve $f(k,\cdot)$ must cross the horizontal axis from below, i.e., at this point $\partial f(k,\cdot)/\partial k > 0$ must hold. Since $\lim_{k\to\infty}(\partial f(k,\cdot)/\partial k) = \lim_{k\to\infty} f(k,\cdot) = \infty$, and $\partial^2 f(k,\cdot)/\partial k^2 > 0$ holds everywhere and is independent of k, there exists a finite k such that $f(k,\cdot) = 0$ holds and a second BGP exists. For a third BGP to exist, $\partial f(k,\cdot)/\partial k < 0$ would have to hold at this point. But this is not possible because of $\partial^2 f(k,\cdot)/\partial k^2 > 0$, implying that no inflection point exists.

If $(\delta+\rho)/\sigma > (\delta+n)$ but $\varphi > (\delta+\rho)/\sigma$, part (ii) is shown as follows. In this case we have $\lim_{k\to 0}(\partial f(k,\cdot)/\partial k) = \infty$, and $\lim_{k\to\infty}(\partial f(k,\cdot)/\partial k) = \lim_{k\to\infty} f(k,\cdot) = \infty$. Further, there is a unique inflection point of $f(k,\cdot)$ given by $k_w = (2-\alpha)(\varphi-(\rho+\delta)/\sigma)/(\varphi(1+\alpha)(-(\delta+n)+(\rho+\delta)/\sigma))$. This demonstrates that the existence of a BGP with endogenous growth implies that there are two BGPs for this case. ■

Proof of Proposition 2

To prove part (i) of Proposition 2, we first note that $det\, J < 0$ is a necessary and sufficient condition for saddle point stability. Knowing that $c^\infty = (k^{1-\alpha} - \varphi k^{2-\alpha} - k(\delta+n) + \varphi k)/(1-\varphi k)$ holds on the BGP, we can calculate $det\, J$ as

$$det\, J = (1-k\varphi)k^{-1}\left(\frac{\varphi(\varphi-(\delta+n))}{(1-k\varphi)^2} - \frac{\alpha(1-\alpha)}{\sigma}k^{-\alpha-1}\right),$$

with k evaluated on the BGP. Further, we know that $f(k,\cdot) = 0$ on the BGP. Dividing $f(k,\cdot)$ by $k^\alpha(1-\varphi k) \neq 0$ we get

$$f_1(k,\cdot) = -\frac{\rho+\delta}{\sigma} + \varphi + \frac{1-\alpha}{\sigma}k^{-\alpha} + (\varphi-(\delta+n))\frac{\varphi k}{1-\varphi k}$$

and $f_1(k,\cdot) = 0$ must hold on the BGP, too. Differentiating $f_1(k,\cdot)$ with respect to k gives

$$\frac{\partial f_1(k,\cdot)}{\partial k} = \frac{\varphi(\varphi-(\delta+n))}{(1-k\varphi)^2} - \frac{\alpha(1-\alpha)}{\sigma}k^{-\alpha-1}.$$

This shows that sign $det\, J = \text{sign}(\partial f_1(k,\cdot)/\partial k)\cdot(1-\varphi k)$.

If the BGP is unique, we have $(\delta+n) - (\rho+\delta)/\sigma \geq 0$. For $\varphi - (\delta+n) > 0$ this gives

$$\lim_{k\to 0} f_1(k,\cdot) = +\infty \text{ and } \lim_{k\to\infty} f_1(k,\cdot) = (\delta+n) - (\rho+\delta)/\sigma \geq 0 \tag{5.1}$$

$$\lim_{k\nearrow\varphi^{-1}} f_1(k,\cdot) = +\infty \text{ and } \lim_{k\searrow\varphi^{-1}} f_1(k,\cdot) = -\infty \tag{5.2}$$

$$\lim_{k\to 0}\frac{\partial f_1(k,\cdot)}{\partial k} = -\infty \text{ and } \lim_{k\to\infty}\partial f_1(k,\cdot) = 0 \tag{5.3}$$

$$\lim_{k\nearrow\varphi^{-1}}\frac{\partial f_1(k,\cdot)}{\partial k} = +\infty \text{ and } \lim_{k\searrow\varphi^{-1}}\frac{\partial f_1(k,\cdot)}{\partial k} = +\infty, \tag{5.4}$$

where $\nearrow$ means that k approaches φ^{-1} from below and $\searrow$ means that k approaches φ^{-1} from above. Since the BGP is unique, (5.1)–(5.4) demonstrate that $f_1(k,\cdot)$ intersects the horizontal axis from below, i.e., $\partial f_1(k,\cdot)/\partial k > 0$ holds at the intersection point, and this point is in the range $k \in (\varphi^{-1},\infty)$. Consequently, $det\, J < 0$ and the rest point is stable in the saddle point sense.

If $(\delta+n) - (\rho+\delta)/\sigma = 0$, $f_1(k,\cdot)$ must intersect the horizontal axis from below and then converge to zero. Since there may exist an inflection point for $f_1(k,\cdot)$ for $\varphi - (\delta+n) > 0$ and $k > \varphi(\cdot)^{-1}$ this possibility is given.

For $\varphi - (\delta + n) < 0$ we have[16]

$$\lim_{k \nearrow \varphi^{-1}} f_1(k, \cdot) = -\infty \text{ and } \lim_{k \searrow \varphi^{-1}} f_1(k, \cdot) = +\infty \tag{5.5}$$

$$\lim_{k \nearrow \varphi^{-1}} \frac{\partial f_1(k, \cdot)}{\partial k} = -\infty \text{ and } \lim_{k \searrow \varphi^{-1}} \frac{{}_1(k, \cdot)}{\partial k} = -\infty, \tag{5.6}$$

whereas (5.1) and (5.3) do not change. (5.5) and (5.6) together with (5.1) and (5.3) show that $f_1(k, \cdot)$ intersects the horizontal axis from above, i.e., $\partial f_1(k, \cdot)/\partial k < 0$ holds, and the intersection point is now in the range $k \in (0, \varphi^{-1})$. Consequently, $det\,J < 0$ and the rest point is again stable in the saddle point sense.

Part (ii) of Proposition 2 is proved as follows. Again we know that $det\,J < 0$ is necessary and sufficient for saddle point stability. Further, the existence of two BGPs implies $(\delta + n) - (\rho + \delta)/\sigma < 0$, and from the proof of part (i) we know that sign $det\,J = \text{sign}(\partial f_1(k, \cdot)/\partial k) \cdot (1 - \varphi k)$.

For $\varphi - (\delta + n) > 0$, $f_1(k, \cdot)$ has the same properties as in the proof of part (i) with $(\delta + n) - (\rho + \delta)/\sigma \geq 0$, with the exception of $\lim_{k \to \infty} f_1(k, \cdot)$, which is now $\lim_{k \to \infty} f_1(k, \cdot) = (\delta + n) - (\rho + \delta)/\sigma < 0$. This together with $\lim_{k \to 0} f_1(k, \cdot) = +\infty$ and (5.2)–(5.4) shows that the values for k on the BGP are either from $k \in (0, \varphi^{-1})$ or from $k \in (\varphi^{-1}, \infty)$.

If the values for k on the BGPs are from $(0, \varphi^{-1})$, we have $1 - \varphi k > 0$, and $f_1(k, \cdot)$ first intersects the horizontal axis from above and then from below, i.e., $\partial f_1(k, \cdot)/\partial k < 0$ holds at the first point and $\partial f_1(k, \cdot)/\partial k > 0$ holds at the second point. This shows that $det\,J < 0$ for the first BGP (with the lower value of k) and $det\,J > 0$ for the second (with the higher value of k).

If the values for k on the BGPs are from (φ^{-1}, ∞), we have $1 - \varphi k < 0$, and $f_1(k, \cdot)$ first intersects the horizontal axis from below and then from above, i.e., $\partial f_1(k, \cdot)/\partial k > 0$ holds at the first point and $\partial f_1(k, \cdot)/\partial k < 0$ at the second point. This shows that $det\,J < 0$ for the first BGP (with the lower value of k) and $det\,J > 0$ for the second (with the higher value of k).

For $\varphi - (\delta + n) < 0$, we get for $f_1(k, \cdot)$

$$\lim_{k \to 0} f_1(k, \cdot) = +\infty \text{ and } \lim_{k \to \infty} f_1(k, \cdot) = (\delta + n) - (\rho + \delta)/\sigma < 0 \tag{5.7}$$

$$\lim_{k \nearrow \varphi^{-1}} f_1(k, \cdot) = -\infty \text{ and } \lim_{k \searrow \varphi^{-1}} f_1(k, \cdot) = +\infty \tag{5.8}$$

$$\lim_{k \to 0} \frac{\partial f_1(k, \cdot)}{\partial k} = -\infty \text{ and } \lim_{k \to \infty} \partial f_1(k, \cdot) = 0 \tag{5.9}$$

$$\lim_{k \nearrow \varphi^{-1}} \frac{\partial f_1(k, \cdot)}{\partial k} = -\infty \text{ and } \lim_{k \searrow \varphi^{-1}} \frac{{}_1(k, \cdot)}{\partial k} = -\infty. \tag{5.10}$$

This shows that for the first BGP (lower k), k is from the range $(0, \varphi^{-1})$ and we have $\partial f_1(k, \cdot)/\partial k < 0$ at the intersection point such that $det\,J < 0$. For the second BGP (higher k), k is from the range (φ^{-1}, ∞) and we also have $\partial f_1(k, \cdot)/\partial k < 0$ at the intersection point, giving $det J > 0$. Thus, Proposition 2 is proved. ■

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[16] Recall that for $\varphi - (\delta + n) = 0$, no BGP with sustained per-capita growth exists.

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