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The Identification of Spurious Lyapunov Exponents in Jacobian Algorithms

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Abstract. *The method of reconstructing an n-dimensional system from observations is to form vectors of m consecutive observations, which for $m > 2n$, is generically an embedding. This is Takens's result. The Jacobian methods for Lyapunov exponents utilize a function of m variables to model the data, and the Jacobian matrix is constructed at each point in the orbit of the data. When embedding occurs at dimension $m = n$, the Lyapunov exponents of the reconstructed dynamics are the Lyapunov exponents of the original dynamics. However, if embedding only occurs for an $m > n$, then the Jacobian method yields m Lyapunov exponents, only n of which are the Lyapunov exponents of the original system. The problem is that as currently used, the Jacobian method is applied to the full m-dimensional space of the reconstruction, and not just to the n-dimensional manifold that is the image of the embedding map. Our examples show that it is possible to obtain* spurious *Lyapunov exponents that are even larger than the largest Lyapunov exponent of the original system.*

Keywords. Lyapunov exponents, embedded dynamics

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1 Introduction

Lyapunov exponents measure the rate of divergence or convergence of two nearby initial points of a dynamical system. A positive Lyapunov exponent measures the average exponential divergence of two nearby trajectories, whereas a negative Lyapunov exponent measures exponential convergence of two nearby trajectories. If a discrete nonlinear system is dissipative, a positive Lyapunov exponent quantifies a measure of chaos.

The introduction of Lyapunov exponents to economics was in Brock (1986). Brock and Sayers (1988) noted that the Wolf (Wolf et al. 1985) algorithm is sensitive to the number of observations as well as to the degree of measurement or system noise in the observations. This discovery motivated a search for new algorithmic designs with improved finite-sample properties. This search for an algorithm to calculate Lyapunov exponents with desirable finite-sample properties has gained momentum in the last few years. Abarbanel et al. (1991a, 1991b, 1992), Ellner et al. (1991), McCaffrey et al. (1992), Gençay and Dechert (1992), and Dechert and Gençay (1996) came up with improved algorithms for calculating the Lyapunov exponents from observed data. Gençay (1994) worked on the calculation of the Lyapunov exponents with noisy data using feedforward networks as the estimation technique.

The main algorithmic design in all of the papers above involves embedding the observations in an m-dimensional space, then employing the theorems of Mañé (1981) and Takens (1981) to use the observations in reconstructing the dynamics on the attractor. The Jacobian of the reconstructed dynamics as demonstrated in Eckmann and Ruelle (1985) and Eckmann et al. (1986) is then used to calculate the Lyapunov exponents of the unknown dynamics. The method of reconstructing an n-dimensional system from observations includes forming vectors of m-consecutive observations, which for $m > 2n$ is generically an embedding process. The Jacobian methods for Lyapunov exponents utilize a function of m variables to model the data, and a Jacobian matrix is constructed at each point in the orbit of the data. When embedding occurs at dimension $m = n$, then the Lyapunov exponents of the reconstructed dynamics are the Lyapunov exponents of the original dynamics. However, if embedding only occurs when $m > n$, then the Jacobian method yields m Lyapunov exponents, only n of which are the Lyapunov exponents of the original system.

The problem is that as it is currently used, the Jacobian method is applied to the full m-dimensional space of the reconstruction, and not just to the n-dimensional manifold that is the image of the embedding map. Our examples show that it is possible to get *spurious* Lyapunov exponents that are even larger than the largest Lyapunov exponent of the original system. Parlitz (1994) focuses on the identification of spurious Lyapunov exponents by presenting a method for experimental data. This method is based on the observation that the true Lyapunov exponents change their signs upon time reversal, whereas the spurious exponents do not. Parlitz's (1994) method can be a useful tool for identification purposes, especially for continuous-time systems. For discrete chaotic systems, in general it is not possible to run time backward, since the dynamics are not one to one.

In Section 2, Lyapunov exponents are defined. The Jacobian algorithm of Lyapunov exponents and its illustration are presented in Sections 3 and 4. Section 5 explains how large spurious Lyapunov exponents may exist, and describes methods for identifying them in practice.

2 Definition of Lyapunov Exponents

The Lyapunov exponents for a dynamical system $f\colon R^n \to R^n$, with the trajectory

$$x_{t+1} = f(x_t) \qquad t = 0, 1, 2, \ldots \tag{1}$$

are measures of the average rate of divergence or convergence of a typical trajectory.[1] For an n-dimensional system as above, there are n exponents which are customarily ranked from largest to smallest:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

Definition 2.1. *Let $f\colon R^n \to R^n$ define a discrete dynamical system, and select a point $x \in R^n$. Let $(Df)_x$ be the matrix of partial derivatives of f evaluated at the point x. Suppose that there are subspaces $R^n = V_t^1 \supset V_t^2 \cdots \supset V_t^{n+1} = \{0\}$ in the tangent space of R^n at $f^t(x)$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that*

(a) $(Df^t)_x(V_t^j) \subseteq V_{t+1}^j$

(b) $\dim V_t^j = n + 1 - j$

(c) $\lambda_j = \lim_{t\to\infty} t^{-1} \ln \|(Df^t)_x v\|$ *for all* $v \in V_0^j \setminus V_0^{j+1}$

then the λ_j are called the Lyapunov exponents of f.

The subspace $V_0^1 \setminus V_0^2$ consists of those vectors which grow at the fastest average rate; $V_0^2 \setminus V_0^3$ consists of those vectors which grow at the next fastest rate, etc.

[1]The trajectory is also written in terms of the iterates of f. With the convention that f^0 is the identity map, and $f^{t+1} = f \circ f^t$, then we also write $x_t = f^t(x_0)$. A trajectory is also called an "orbit" in the dynamical system literature.

It is a consequence of Oseledec's (1968) theorem that the Lyapunov exponents exist for a broad class of functions.[2] The additional properties of Lyapunov exponents and a formal definition[3] are given in Guckenheimer and Holmes (1983). Notice that for $j \geq 2$ the subspaces V^j are sets of Lebesgue measure zero, and so for almost all $v \in R^n$, the limit in part (c) of Definition 2.1 equals λ_1. This is the basis for the computational algorithm of Wolf et al. (1985), which is a method for calculating the largest Lyapunov exponent.

3 The Jacobian Algorithm

In practice, one rarely has the advantage of observing the state of the system, x_t, let alone knowing the actual functional form f which generates the dynamics. The model that is widely used is the following: associated with the dynamical system in Equation (1), there is an observer function $h: R^n \to R$ which generates the observations

$$y_t = h(x_t)$$

It is assumed that all that is available to the researcher is the sequence $\{y_t\}$. For notational purposes, let

$$y_t^m = (y_t, y_{t+1}, \ldots, y_{t+m-1}) \tag{2}$$

If the set $\overline{U}$ is compact manifold, then for $m \geq 2n+1$

$$J^m(x) = \big(h(x), h(f(x)), \ldots, h(f^{m-1}(x))\big) \tag{3}$$

generically is an embedding.[4] For $m \geq 2n+1$, there exists a function $g: R^m \to R^m$ such that

$$y_{t+1}^m = g(y_t^m)$$

where

$$y_{t+1}^m = (y_{t+1}, y_{t+2}, \ldots, y_{t+m})$$

But notice that

$$y_{t+1}^m = J^m(x_{t+1}) = J^m(f(x_t)) \tag{4}$$

Hence, from Equations (2) and (4),

$$J^m(f(x_t)) = g(J^m(x_t))$$

The function g is topologically conjugate to f. This implies that g inherits the dynamical properties of f. Dechert and Gençay (1996) prove the following theorem to show that n of the Lyapunov exponents of g are the Lyapunov exponents of f.

Theorem 3.1 (Dechert and Gençay 1996). *Assume that M is a smooth manifold dimension n, $f: M \to M$ and $h: M \to R$ are (at least) C^2. Define $J^m: M \to R^m$ by $J^m(x) = \big(h(x), h(f(x)), \ldots, h(f^{m-1}(x))\big)$. Let $\mu_1(x) \geq \mu_2(x) \geq \cdots \geq \mu_n(x)$ be the eigenvalues of the symmetric matrix $(DJ^m)'_x(DJ^m)_x$, and suppose that*

$$\inf_{x\in M} \mu_n(x) > 0$$
$$\sup_{x\in M} \mu_1(x) < \infty$$

Let $\lambda_1^f \geq \lambda_2^f \geq \cdots \geq \lambda_n^f$ be the Lyapunov exponents of f and $\lambda_1^g \geq \lambda_2^g \geq \cdots \geq \lambda_m^g$ be the Lyapunov exponents of g, where $g: J^m(M) \to J^m(M)$ and $J^m(f(x)) = g(J^m(x))$ on M. Then generically $\{\lambda_i^f\} \subset \{\lambda_i^g\}$.

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[2] Also see Cohen et al. (1986), Raghunathan (1979), and Ruelle (1979) for precise conditions and proofs of the theorem.

[3] Definition 2.1 differs slightly from Guckenheimer and Holmes (1983) in that we use set containment rather than set equality in part (a) of the definition. When equality holds in part (a), we say that the dynamics are of full dimension.

[4] By *generic* we mean that in every neighborhood of f and h, there are functions $\tilde{f}$ and $\tilde{h}$ such that the function J^m corresponding to these functions is an embedding of the attractor of $\tilde{f}$ and the image of the attractor under J^m. Here, $2n+1$ is the worst-case upper limit.

By Theorem 3.1, n of the Lyapunov exponents of g are the Lyapunov exponents of f. The approach of Gençay and Dechert (1992) is to estimate the function g based on the data sequence $\{J^m(x_t)\}$, and to calculate the Lyapunov exponents of g. The identification of the n Lyapunov exponents of f from the m Lyapunov exponents of g is discussed in Section 5.

From Equation (2), the map g which is to be estimated may be taken[5] to be

$$g : \begin{bmatrix} y_t \\ y_{t+1} \\ \vdots \\ y_{t+m-1} \end{bmatrix} \to \begin{bmatrix} y_{t+1} \\ y_{t+2} \\ \vdots \\ v(y_t, y_{t+1}, \ldots, y_{t+m-1}) \end{bmatrix} \tag{5}$$

and this reduces to estimating

$$y_{t+m} = v(y_t, y_{t+1}, \ldots, y_{t+m-1})$$

Here v is an unknown map. Linearization of the map g yields

$$\Delta y^m_{t+1} = (Dg)_{y^m_t} \Delta y^m_t$$

The solution can be written as

$$\Delta y^m_t = (Dg^t)_{y^m_0} \Delta y^m_0$$

The Lyapunov exponents can be calculated from the eigenvalues of the matrix $(Dg^t)_{y^m_0}$ using QR decomposition. This method is discussed in Eckmann and Ruelle (1985), Eckmann et al. (1986), and Sano and Sawada (1985), and a modified version is presented in Abarbanel et al. (1992).

4 An Illustration

If x is a fixed point, then the subspaces $V^j_t = V^j$ do not depend upon t. Let us consider the mapping $f(x)$ at the fixed point x. Choose $V^1 = R^2$, $V^2 = \text{span}\{(0, 1)\}$, and $V^3 = \{0\}$. For $|\mu_1| > |\mu_2|$ consider[6]

$$Df(x) = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \tag{6}$$

This will satisfy parts (a) and (b) of Definition 2.1, and we will have

$$\begin{aligned} \lambda_1 &= \lim_{t\to\infty} t^{-1} \ln(|\mu_1^t v_1 + \mu_2^t v_2|) = \ln|\mu_1| \quad \textit{for} \quad v \in V^1 \setminus V^2 \\ \lambda_2 &= \lim_{t\to\infty} t^{-1} \ln(|\mu_1^t v_1 + \mu_2^t v_2|) = \ln|\mu_2| \quad \textit{for} \quad v \in V^2 \setminus V^3. \end{aligned}$$

This definition primarily generalizes the idea of eigenvalues to give average linearized contraction and expansion rates on a trajectory.

An attractor is a set of points toward which the trajectories of f converge. More precisely, Λ is an attractor if there is an open set $U \subset R^n$ with $\Lambda \subset U$, $f(\overline{U}) \subset U$, and

$$\Lambda = \bigcap_{t\geq 0} f^t(U)$$

where $\overline{U}$ is the closure of U. The attractor Λ is said to be indecomposable if there is no proper subset of Λ which is also an attractor. An attractor can be chaotic or ordinary (or nonchaotic). There is more than one definition of a chaotic attractor in the literature. In practice, the presence of a positive Lyapunov exponent is taken as a signal that the attractor is chaotic.

[5]Here, the time step is assumed to be equal to the delay time.

[6]This example is from Guckenheimer and Holmes (1983).

Now, suppose that the observations come from the following:

$$y = h(x) = x_1 + x_2 \tag{7}$$

where $h: R^2 \to R$. Let us consider a three-embedding history generated from $h(x)$ so that

$$J^3(x) = \begin{bmatrix} 1 & 1 \\ \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \end{bmatrix} x \tag{8}$$

and

$$J^3 \circ f(x) = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \\ \mu_1^3 & \mu_2^3 \end{bmatrix} x \tag{9}$$

Let

$$g(y) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\mu_1\mu_2 & \mu_1 + \mu_2 \end{bmatrix} y$$

for $y \in R^3$. Then,

$$g \circ J^3(x) = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \\ \mu_1^3 & \mu_2^3 \end{bmatrix} x = J^3 \circ f(x)$$

Therefore, the condition for conjugacy is satisfied. Also,

$$(Dg)_y = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\mu_1\mu_2 & \mu_1 + \mu_2 \end{bmatrix} \tag{10}$$

Let $W^1 = R^3$, $W^2 = \text{span}\{(1,0,0), (1,\mu_2,\mu_2^2)\}$, $W^3 = \text{span}\{(1,0,0)\}$, and $W^4 = \{0\}$. Then,

$$\begin{aligned} (Dg)_y(W^1) &= \text{span}\{(1,\mu_1,\mu_1^2), (1,\mu_2,\mu_2^2)\} \subset W^1 \\ (Dg)_y(W^2) &= \text{span}\{(1,\mu_2,\mu_2^2)\} \subset W^2 \end{aligned}$$

and

$$(Dg)_y(W^3) = \{0\} \subset W^3$$

Notice that the sets $(Dg)_y W^j$ can be proper subsets of W^j (see footnote 3). In this example, this comes about since the dynamics of g are not of full dimension, which is immediately apparent from Equation (10). If $v \in V^1 \setminus V^2$ then

$$v = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \alpha \neq 0$$

and

$$(DJ^3)v = \alpha \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix}$$

Here, $\alpha \neq 0$ implies that $(DJ^3)v \in W^1 \setminus W^2$. If $v \in V^2 \setminus V^3$ then

$$v = \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \beta \neq 0$$

and

$$(DJ^3)v = \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix}$$

Also $\beta \neq 0$ implies that $(DJ^3)v \in W^2 \setminus W^3$. If $w \in W^1 \setminus W^2$, then

$$w = \alpha \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \alpha \neq 0$$

and

$$|(Dg)_y^t w| = \left| \alpha \mu_1^t \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \mu_2^t \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} \right|$$

Hence, $\lim_{t\to\infty} t^{-1} \ln |(Dg^t)_y w| = \ln |\mu_1|$.

If $w \in W^2 \setminus W^3$, then

$$w = \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \beta \neq 0$$

and

$$|(Dg^t)_y w| = \left| \beta \mu_2^t \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} \right|$$

Hence, $\lim_{t\to\infty} t^{-1} \ln |(Dg^t)_y w| = \ln |\mu_2|$.

If $w \in W^3 \setminus W^4$ then

$$w = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \gamma \neq 0$$

and $|(Dg^t)_y w| = 0$. Therefore $\lim_{t\to\infty} t^{-1} \ln |(Dg)_y^t w| = -\infty$. This example shows Theorem 3.1 at work. The two largest Lyapunov exponents of g are the Lyapunov exponents of f, and in this example the "spurious" third exponent of g is $-\infty$.

5 Spurious Lyapunov Exponents

In Dechert and Gençay (1992) and Gençay and Dechert (1992), the numerical studies demonstrated that the n Lyapunov exponents of f turned out to be the largest n Lyapunov exponents of g. These results were obtained by using an observation function of the form:

$$h(x_1, x_2, \ldots, x_n) = x_1 \tag{11}$$

which has been widely used in simulation studies of nonlinear dynamical systems.

Consider the following variation to the example in the previous section. The dynamics are the same linear dynamics of Equation (6), and the observation function is the same as Equation (7). From this, we obtain the same embedding equations as Equations (8) and (9). Now however, consider the following function g: for any $a \in R$, let

$$g(y) = \begin{bmatrix} a & 1 - a(\mu_1^{-1} + \mu_2^{-1}) & a\mu_1^{-1}\mu_2^{-1} \\ 0 & 0 & 1 \\ 0 & -\mu_1\mu_2 & \mu_1 + \mu_2 \end{bmatrix} y \tag{12}$$

for $y \in R^3$. Notice that this is not in the form of Equation (5); however, it does satisfy

$$g \circ J^3(x) = \begin{bmatrix} \mu_1 & \mu_2 \\ \mu_1^2 & \mu_2^2 \\ \mu_1^3 & \mu_2^3 \end{bmatrix} x = J^3 \circ f(x)$$

and therefore the condition for conjugacy is satisfied.[7] Also,

$$(Dg)_y = \begin{bmatrix} a & 1 - a(\mu_1^{-1} + \mu_2^{-1}) & a\mu_1^{-1}\mu_2^{-1} \\ 0 & 0 & 1 \\ 0 & -\mu_1\mu_2 & \mu_1 + \mu_2 \end{bmatrix}$$

If $|\mu_2| > |a|$, let $W^1 = R^3$, $W^2 = \text{span}\{(1, 0, 0), (1, \mu_2, \mu_2^2)\}$, $W^3 = \text{span}\{(1, 0, 0)\}$, and $W^4 = \{0\}$. Then if $a = 0$,

$$\begin{aligned} (Dg)_y(W^1) &= \text{span}\{(1, \mu_1, \mu_1^2), (1, \mu_2, \mu_2^2)\} \subset W^1 \\ (Dg)_y(W^2) &= \text{span}\{(1, \mu_2, \mu_2^2)\} \subset W^2 \end{aligned}$$

and

$$(Dg)_y(W^3) = \{0\} \subset W^3$$

If $a \neq 0$, then

$$\begin{aligned} (Dg)_y(W^1) &= W^1 \\ (Dg)_y(W^2) &= W^2 \end{aligned}$$

and

$$(Dg)_y(W^3) = W^3$$

If $v \in V^1 \setminus V^2$, then

$$v = \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \alpha \neq 0$$

and

$$(DJ^3)v = \alpha \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix}$$

Here, $\alpha \neq 0$ implies that $(DJ^3)v \in W^1 \setminus W^2$. If $v \in V^2 \setminus V^3$, then

$$v = \beta \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \beta \neq 0$$

and

$$(DJ^3)v = \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix}$$

Also, $\beta \neq 0$ implies that $(DJ^3)v \in W^2 \setminus W^3$. If $w \in W^1 \setminus W^2$, then

$$w = \alpha \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \alpha \neq 0$$

[7] This shows that there can be many functions that can generate the same dynamics. In our case, we are interested in the impact that the observer function has on this multiplicity of representations, g.

and

$$|(Dg^t)_y w| = \left| \alpha \mu_1^t \begin{bmatrix} 1 \\ \mu_1 \\ \mu_1^2 \end{bmatrix} + \beta \mu_2^t \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma a^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right|$$

Hence, $\lim_{t\to\infty} t^{-1} \ln |(Dg^t)_y w| = \ln |\mu_1|$.

If $w \in W^2 \setminus W^3$, then

$$w = \beta \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \beta \neq 0$$

and

$$|(Dg^t)_y w| = \left| \beta \mu_2^t \begin{bmatrix} 1 \\ \mu_2 \\ \mu_2^2 \end{bmatrix} + \gamma a^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right|$$

Hence, $\lim_{t\to\infty} t^{-1} \ln |(Dg^t)_y w| = \ln |\mu_2|$.

If $w \in W^3 \setminus W^4$, then

$$w = \gamma \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \gamma \neq 0$$

and $|(Dg^t)_y w| = |\gamma||a|^t$. Therefore, $\lim_{t\to\infty} t^{-1} \ln |(Dg)_y^t w| = \ln |a|$. Note that if $a = 0$, then this third "spurious" Lyapunov exponent is $-\infty$.

If $|\mu_1| > |a| > |\mu_2|$, then the subspace W^3 above needs to be changed so that $W^3 = \text{span}\{(1, \mu_2, \mu_2^2)\}$. Then

$$\begin{aligned} (Dg)_y(W^1) &= W^1 \\ (Dg)_y(W^2) &= W^2 \end{aligned}$$

and

$$(Dg)_y(W^3) = W^3$$

The three Lyapunov exponents are $\ln |\mu_1|$, $\ln |a|$, and $\ln |\mu_2|$. If $|a| > |\mu_1|$, we change the subspaces so that $W^2 = \text{span}\{(1, \mu_1, \mu_1^2), (1, \mu_2, \mu_2^2)\}$, $W^3 = \text{span}\{(1, \mu_2, \mu_2^2)\}$, and again

$$\begin{aligned} (Dg)_y(W^1) &= W^1 \\ (Dg)_y(W^2) &= W^2 \end{aligned}$$

and

$$(Dg)_y(W^3) = W^3$$

will hold. The three Lyapunov exponents are then $\ln |a|$, $\ln |\mu_1|$, and $\ln |\mu_2|$.

Notice that in all cases, the two Lyapunov exponents of f are two of the Lyapunov exponents of g. The third Lyapunov exponent of g can be of any magnitude. The problem emerges because the partial derivatives of g do not necessarily lie in the tangent space of the image of the attractor under the Takens embedding (Equation [3]). It raises the question of how to identify the n *true* Lyapunov exponents of f from the $m - n$ *spurious* Lyapunov exponents that make up the Lyapunov exponents of g.

From a theoretical point of view, the answer lies in the fact that the derivatives (and, therefore, the Lyapunov exponents) of g are only defined on the tangent space of the image of the manifold under the Takens embedding J^m. The dimension of this tangent space is the same as the dimension of the manifold, so there are precisely n derivatives of g in the tangent space of the image of the manifold. However, when an m-dimensional function is fitted to the data, no matter how good the fit is on the data, the fitted function will have m derivatives, and hence m Lyapunov exponents. In theory, it is an open question of how to identify the Lyapunov exponents of an unknown system.

In practice, it is necessary to identify the tangent spaces to the attractor in the embedding space at each observation. One solution is to follow the method in Eckmann and Ruelle (1985) as a second diagnostic to identify the spurious Lyapunov exponents. To identify the largest Lyapunov exponent, estimate the degree of spreading of nearby vectors. To identify the sum of the two largest Lyapunov exponents, estimate the degree of spreading of nearby plane areas (as determined by pairs of vectors). This process can be repeated to identify all of the Lyapunov exponents.

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The *SNDE* is formed in recognition that advances in statistics and dynamical systems theory may increase our understanding of economic and financial markets. The journal will seek both theoretical and applied papers that characterize and motivate nonlinear phenomena. Researchers will be encouraged to assist replication of empirical results by providing copies of data and programs online. Algorithms and rapid communications will also be published.

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