



Studies in Nonlinear Dynamics and Econometrics

Quarterly Journal

July 1998, Volume 3, Number 2

The MIT Press

Studies in Nonlinear Dynamics and Econometrics (ISSN 1081-1826) is a quarterly journal published electronically on the Internet by The MIT Press, Cambridge, Massachusetts, 02142. Subscriptions and address changes should be addressed to MIT Press Journals, Five Cambridge Center, Cambridge, MA 02142; tel.: (617) 253-2889; fax: (617) 577-1545; e-mail: journals-orders@mit.edu. Subscription rates are: Individuals \$40.00, Institutions \$130.00. Canadians add 7% GST. Prices subject to change without notice.

Subscribers are licensed to use journal articles in a variety of ways, limited only as required to insure fair attribution to authors and the Journal, and to prohibit use in a competing commercial product. See the Journal's World Wide Web site for further details. Address inquiries to the Subsidiary Rights Manager, MIT Press Journals, Five Cambridge Center, Cambridge, MA 02142; tel.: (617) 253-2864; fax: (617) 258-5028; e-mail: journals-rights@mit.edu.

Smooth-Transition GARCH Models

Gloria González-Rivera
Department of Economics
University of California, Riverside
Riverside, California 92521, U.S.A.
ggonzale@galaxy.ucr.edu

Abstract. *The asymmetric response of conditional variances to positive versus negative news has been traditionally modeled with threshold specifications that allow only two possible regimes: low or high volatility. In this paper, the possibility of intermediate regimes is considered and modeled with the introduction of a smooth-transition mechanism in a GARCH specification. One important property of this model is that it permits an on-off ARCH effect, in which a time series can switch from a process with constant variance to a process with time-varying variance. On testing for the existence of a smooth-transition mechanism, there are nuisance parameters that are not identified under the null hypothesis. Nevertheless, it is possible to construct a Lagrange-multiplier test that is χ_p^2 -distributed. A Monte Carlo simulation shows that the test has very good size and good power. A smooth-transition GARCH specification is tested and estimated with stock returns and exchange-rate data. While a threshold model is preferred for stock returns, a smooth-transition model is more likely for exchange rates.*

Keywords. GARCH; leverage effect; news-impact curve; smooth transition; threshold

Acknowledgments. I wish to thank to the participants of the 1997 UCSD-UCR Conference on Time Series Analysis of High Frequency Financial Data, the 4th Annual Conference of the Multinational Finance Society, and seminars at UCSD, UCR, Universidad Carlos III of Madrid, Universidad Complutense of Madrid, Tilburg University, Arizona State University, and UC Berkeley. Special thanks to Timo Teräsvirta for his comments on a careful reading of the manuscript. Financial support from the Academic Senate of the University of California, Riverside, is gratefully acknowledged. The usual caveat applies.

1 Introduction

A significant feature encountered in the modeling of financial data is the asymmetric response of the volatility process to unanticipated shocks. It is found that financial markets become more volatile in response to negative shocks than to positive shocks. While the economic reasons behind this behavior are not well understood (Black 1976; Christie 1982; Schwert 1990), the econometric modeling of this asymmetry, known as the leverage effect, has produced quite a number of ARCH models (see Nelson 1990; Glosten, Jagannathan, and Runkle, or GJR 1993; Zakoian 1994; Engle 1990; Engle and Ng 1993; Ding, Granger, and Engle, or DGE 1993).

The differences among these models reside in the behavior of the news-impact curve (Engle and Ng 1993). This curve relates past innovations (news), ε_{t-1} , to current volatility, and may have either different slopes for positive and negative ε_{t-1} values, or the curve minimum may be shifted toward the positive ε_{t-1} values. Models with different slopes are found in the works of Nelson (1990), Glosten, Jagannathan, and Runkle (1993), Zakoian (1994), and Ding, Granger, and Engle (1993). News-impact curves with a shifted minimum are found in models by Engle (1990) and Engle and Ng (1993).

However, the common characteristic shared by all of these models is the existence of only two regimes: low and high volatilities, which are triggered by positive and negative shocks, respectively. In this sense, all are threshold models where the threshold is known and is equal to zero. Furthermore, these models are habitually applied to stock returns. Recent articles by Liu, Li, and Li (1997) and Li and Li (1996) have considered ARCH specifications with multiple thresholds, but in their empirical applications to stock returns there is only one threshold and it is equal to zero.

This paper generalizes the modeling of asymmetry in variance with the introduction of a smooth-transition specification for conditional variances. In addition, it expands the application of asymmetric models to exchange-rate data. The smooth transition is an extension of the two-regime variance, since it allows intermediate states or regimes. It also nests a threshold specification, since for certain parameter values, smooth-transition models collapse to threshold models.

In a closely related literature, Markov switching models deal with the changes in regime in GARCH specifications. Articles by Hamilton and Susmel (1994), Cai (1994), and Dueker (1997) discuss how the conditional variances may come from a discrete number of regimes with the transition between regimes governed by an unobserved Markov chain. Theoretically, the number of regimes or states can be large, but practical considerations require a reduced number of states. In addition, there is a probability-transition matrix that must be estimated. The framework of the present paper is very different from that of Markov-switching models. In the context of smooth-transition models, I use the word *regime* with a different meaning from that of a Markov model. I specify a transition function that is continuous and may exhibit different degrees of smoothness. Because this function is continuous, we may talk of a *continuum* of regimes where the probability of jumping from one regime to another is one. The degree of smoothness also controls the number of regimes. A sharp transition function indicates two regimes, high versus low volatility, whereas a smoother transition function allows for other, intermediate regimes. The moving among regimes is dictated by an observable transition variable that belongs to the history of the process.

Testing for the existence of a smooth-transition mechanism in GARCH models presents similar problems to those encountered in the smooth-transition autoregressive (STAR) models (Teräsvirta 1994; Lukkonen, Saikkonen, and Teräsvirta 1988). There are several ways to formulate the null hypothesis of interest. This is equivalent to saying that parameter identification is an issue. Under the null hypothesis, there are nuisance parameters that are not identified; they exist only under the alternative. Since these parameters cannot be estimated under the null hypothesis, the standard asymptotic theory does not apply. A comprehensive discussion of these testing issues can be found in the work of Andrews and Ploberger (1994). Nevertheless, in the context of this paper, the nonidentification problem can be resolved, and a standard Lagrange-multiplier (LM) test can be constructed that is χ^2 -distributed. The solution depends on the specification of the nonidentifiable parameters. If the specification is linear, it is possible to construct a standard LM test; if the specification is nonlinear, the LM test has a nonstandard distribution.

Asymmetric models in variance have been applied only to stock returns. In this paper, I also consider exchange-rate data. If a threshold model is estimated for exchange rates, the evidence of asymmetry is very weak. This may explain why this sample has not been a popular choice for use with threshold models. However, if a smooth-transition model is estimated, the evidence of asymmetry becomes stronger. It turns out that the transition from a low- to a high-volatility regime is smooth, rejecting the threshold specification where the transition is sharp.

The organization of this paper is as follows. In Section 2, I introduce the smooth-transition model and its characteristics. In Section 3, I address the problem of testing for the existence of a smooth-transition mechanism. In Section 4, I offer an application to stock returns and exchange rates, and in Section 5, I conclude and summarize this work.

2 The Model

Let ε_t be a random variable, ψ_{t-1} be the information set containing information up to time $t - 1$, and h_t be the conditional variance of ε_t . Suppose that ε_t is governed by

$$\varepsilon_t = u_t \sqrt{h_t},$$

where u_t is an i.i.d. sequence with zero mean and unit variance. I assume that ε_t is distributed conditionally normal,

$$\varepsilon_t | \psi_{t-1} \rightarrow N(0, b_t).$$

The distributional assumption may be relaxed to a more general set of assumptions. If conditional normality does not hold, a quasi-maximum-likelihood estimation (QMLE) framework will apply.

Definition 1. A smooth-transition GARCH(p, q, d), called an ST-GARCH, is defined by the model

$$b_t = \omega + \sum_{i=1}^p \alpha_{1i} \varepsilon_{t-i}^2 + \left(\sum_{i=1}^p \alpha_{2i} \varepsilon_{t-i}^2 \right) F(\varepsilon_{t-d}, \gamma) + \sum_{i=1}^q \beta_i b_{t-i}, \quad (1)$$

where

$$F(\varepsilon_{t-d}, \gamma) = \frac{1}{1 + e^{\gamma \varepsilon_{t-d}}} - \frac{1}{2}, \quad \gamma > 0,$$

and where ε_{t-d} is the transition variable, $d \leq p$, and γ is the smoothness parameter.

Considering that the F function is bounded, $-1/2 < F(\varepsilon_{t-d}, \gamma) < 1/2$, sufficient conditions to ensure strictly positive conditional variances are $\omega > 0$, $\alpha_{1i} \geq 0$, $\alpha_{1i} \geq \frac{1}{2} |\alpha_{2i}|$ for $i = 1 \dots p$, and $\beta_i \geq 0$ for $i = 1 \dots q$. Note that α_{2i} can be positive or negative, but since my interest is to capture higher volatility for negative news than for positive news, the parameters α_{2i} and γ need to have the same sign. I have assumed a positive γ , so consequently α_{2i} should also be positive.

In nonlinear models, it is customary to analyze dynamics by examining the stationarity properties of the limiting processes. Following Bollerslev (1986) and Milhøj (1985), in the upper regime, $F(-\infty, \gamma) = 1/2$, and the process is covariance stationary if and only if

$$\sum_{i=1}^p \alpha_{1i} + \frac{1}{2} \sum_{i=1}^p \alpha_{2i} + \sum_{i=1}^q \beta_i < 1.$$

In the lower regime, $F(+\infty, \gamma) = -1/2$, and the process is covariance stationary if and only if

$$\sum_{i=1}^p \alpha_{1i} - \frac{1}{2} \sum_{i=1}^p \alpha_{2i} + \sum_{i=1}^q \beta_i < 1.$$

Similar conditions can be found for any other intermediate regime: for instance, in the mid-regime $F(0, \gamma) = 0$, the process is covariance stationary if and only if

$$\sum_{i=1}^p \alpha_{1i} + \sum_{i=1}^q \beta_i < 1.$$

Note that because $\alpha_{2i} > 0$, covariance stationarity of the upper regime implies covariance stationarity in any other regime, but not vice versa. If $\alpha_{2i} < 0$, the opposite is true, and covariance stationarity of the lower regime implies covariance stationarity of the upper regime.

I will focus on the ST-GARCH(1,1,1) specification to facilitate the explanation of the relevant characteristics of the smooth-transition model:

$$b_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-1}^2 F(\varepsilon_{t-1}, \gamma) + \beta b_{t-1}. \quad (2)$$

2.1 The asymmetric response of b_t

In Equation 2, ε_{t-1} is the transition variable. The function F is monotonically decreasing, $-1/2 < F(\varepsilon_{t-1}, \gamma) < 1/2$, with asymptotes $F(-\infty, \gamma) = 1/2$ and $F(+\infty, \gamma) = -1/2$. For $\varepsilon_{t-1} \leq 0$, $0 \leq F(\varepsilon_{t-1}, \gamma) \leq 1/2$; for $\varepsilon_{t-1} \geq 0$, $-1/2 < F(\varepsilon_{t-1}, \gamma) \leq 0$. Consequently, for $\alpha_2 > 0$, negative news produces higher volatility than positive news.

2.2 Limiting processes of h_t

In a smooth-transition GARCH model, the volatility process can switch between different regimes according to the values of the function F . The switching is constrained by the following limiting processes. When $\varepsilon_{t-1} \rightarrow +\infty$, the lower regime is

$$h_t = \omega + \left(\alpha_1 - \frac{1}{2}\alpha_2 \right) \varepsilon_{t-1}^2 + \beta h_{t-1}, \quad (3)$$

with positive constraint $(\alpha_1 - \frac{1}{2}\alpha_2) \geq 0$.

When $\varepsilon_{t-1} \rightarrow -\infty$, the upper regime is

$$h_t = \omega + \left(\alpha_1 + \frac{1}{2}\alpha_2 \right) \varepsilon_{t-1}^2 + \beta h_{t-1}. \quad (4)$$

Intermediate values of ε_{t-1} give rise to a process for h_t that is a mixture of the limiting regimes.

It is interesting to note that this smooth-transition specification of the conditional variance may produce an on-off ARCH effect. If $\alpha_1 = \alpha_2/2$, the lower regime does not have time-varying conditional variances. Let us call $\lambda \equiv \alpha_1 - \alpha_2/2$. By backward substitution in Equation 3,

$$h_t = \omega(1 + \beta + \beta^2 + \dots) + \lambda(\varepsilon_{t-1}^2 + \beta\varepsilon_{t-2}^2 + \beta^2\varepsilon_{t-3}^2 + \dots);$$

when $\lambda = 0$, h_t converges to a constant, and for $\beta < 1$,

$$h_t = \frac{\omega}{1 - \beta}.$$

The process of ε_t is bouncing between a process with constant variance for positive news and a process with time-varying conditional variance for negative news.

Figure 1 displays an ST-GARCH(1,1,1) process with an on-off ARCH effect. A sample path is simulated for a sample size of 500 observations and parameters $\omega = 0.2$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta = 0.6$, and $\gamma = 1.0$. The upper and lower regimes are also plotted.

2.3 Convergence to the threshold model

If the smoothness parameter, γ , is large, the function F becomes steep. In this case, the smooth-transition model is not distinguishable from the threshold model, because there are only two possible regimes: one for bad news (negative ε 's) and the other for good news (positive ε s), with the threshold set to zero.

Figure 2 displays the shape of the function F for several values of γ .

The equivalence between an ST-GARCH with a large γ and a threshold GARCH model is readily seen using the DGE model. I have chosen the DGE model because it encompasses a big array of ARCH specifications, among them the GJR and the Zakoian models. Furthermore, in the empirical analysis carried out by Engle and Ng (1993), the GJR model outperformed all the other models that take into account asymmetries in volatility. The asymmetric power model of DGE with power equal to 2 is:

$$\begin{aligned} h_t &= \omega + \alpha(|\varepsilon_{t-1}| - \delta\varepsilon_{t-1})^2 + \beta h_{t-1} \\ &= \omega + \alpha(1 + \delta^2)\varepsilon_{t-1}^2 - 2\alpha\delta|\varepsilon_{t-1}|\varepsilon_{t-1} + \beta h_{t-1}. \end{aligned}$$

For negative shocks, where $\varepsilon_{t-1} < 0$,

$$\begin{aligned} \text{DGE:} & & h_t &= \omega + \alpha(1 + \delta)^2\varepsilon_{t-1}^2 + \beta h_{t-1}, \\ \text{ST-GARCH (large } \gamma\text{):} & & h_t &= \omega + (\alpha_1 + \frac{1}{2}\alpha_2)\varepsilon_{t-1}^2 + \beta h_{t-1}. \end{aligned}$$

For positive shocks, where $\varepsilon_{t-1} > 0$,

$$\begin{aligned} \text{DGE:} & & h_t &= \omega + \alpha(1 - \delta)^2\varepsilon_{t-1}^2 + \beta h_{t-1}, \\ \text{ST-GARCH (large } \gamma\text{):} & & h_t &= \omega + (\alpha_1 - \frac{1}{2}\alpha_2)\varepsilon_{t-1}^2 + \beta h_{t-1}. \end{aligned}$$

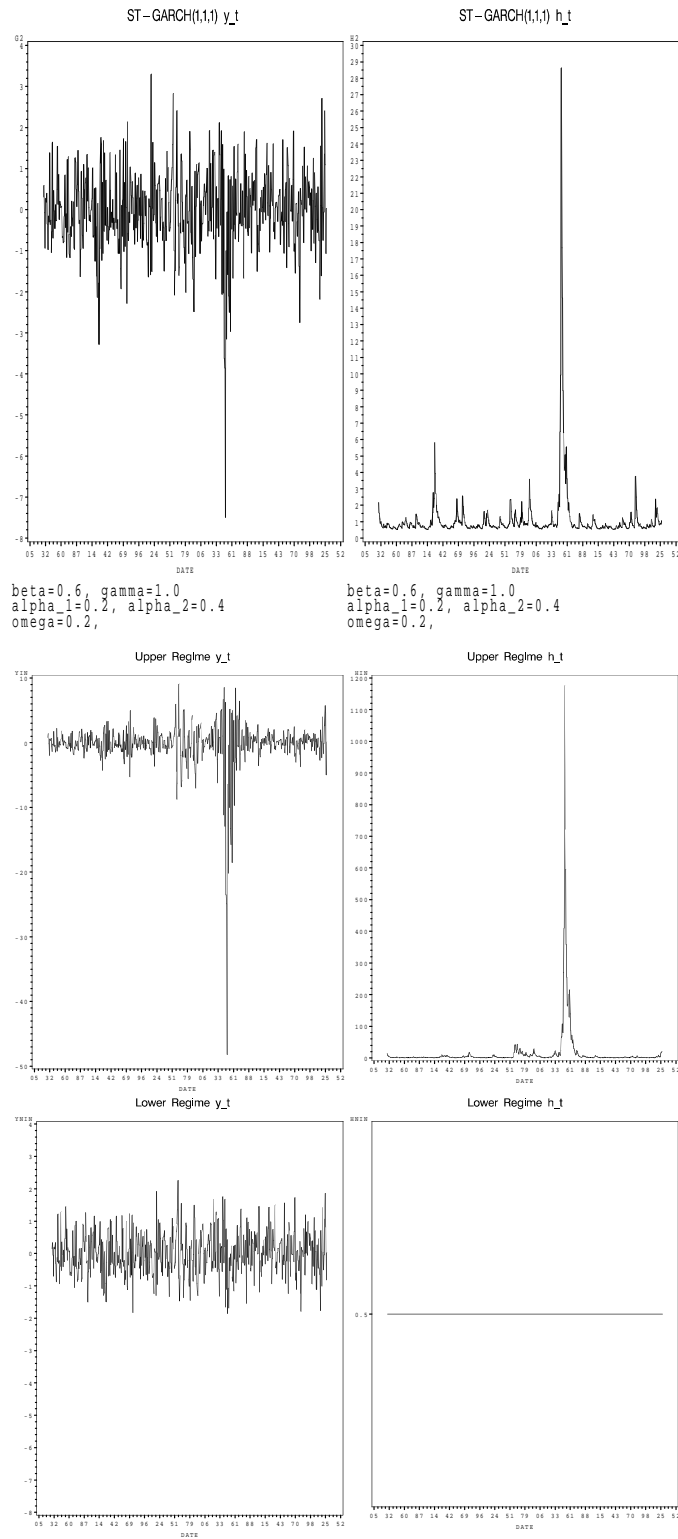


Figure 1

The simulated ST-GARCH(1,1,1) and corresponding upper and lower regimes. Parameters: $\omega = 0.2$, $\alpha_1 = 0.2$, $\alpha_2 = 0.4$, $\beta = 0.6$, and $\gamma = 1.0$.

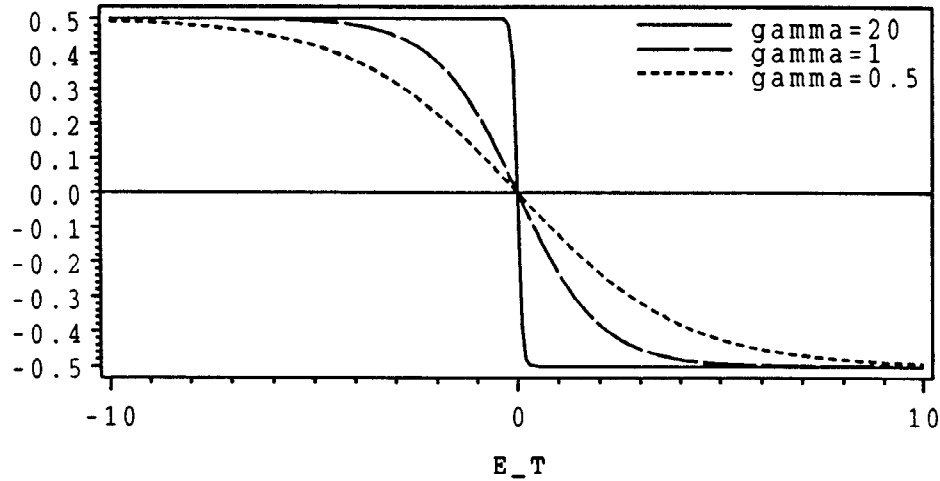


Figure 2
The shape of the function F for three values of γ .

If the probability density function of ε_t is symmetric, the equivalence between the ST-GARCH model and the threshold model is given by the restrictions

$$\alpha_1 = \alpha(1 + \delta^2) \quad \text{and} \quad \alpha_2 = 4\alpha\delta,$$

which constitute the solution to the following system of equations:

$$\alpha(1 + \delta)^2 = \alpha_1 + \frac{1}{2}\alpha_2,$$

$$\alpha(1 - \delta)^2 = \alpha_1 - \frac{1}{2}\alpha_2.$$

2.4 Moments and the news-impact curve

The unconditional variance of ε_t is obtained by applying unconditional expectation to h_t in Equation 2. For symmetric densities, $E(\varepsilon_{t-1}^2 F(\varepsilon_{t-1}, \gamma)) = 0$, because it is the expectation of an odd function.¹ Consequently, the asymmetry term does not affect the unconditional variance σ^2 ,

$$\sigma^2 = \frac{\omega}{1 - \alpha_1 - \beta}.$$

The persistence parameter is given by $\alpha_1 + \beta$, and the condition for ε_t to be covariance stationary is $\alpha_1 + \beta < 1$, as in the GARCH(1,1) of Bollerslev (1986).

The unconditional fourth moment of ε_t is affected by the asymmetry term, since $E(\varepsilon_{t-1}^4 F^2(\varepsilon_{t-1}, \gamma)) \neq 0$ regardless of the symmetry or asymmetry of the conditional probability density function of ε_t . The unconditional fourth moment is bounded by the unconditional fourth moment of the limiting processes. For the lower regime in Equation 3,

$$E(\varepsilon_t^4) = \frac{3\omega^2(1 + \alpha_1 - 0.5\alpha_2 + \beta)}{(1 - \alpha_1 + 0.5\alpha_2 - \beta)(1 - \beta^2 - 2(\alpha_1 - 0.5\alpha_2)\beta - 3(\alpha_1 - 0.5\alpha_2)^2)}.$$

¹For asymmetric densities, the following integral must be solved to obtain the unconditional variance:

$$E_{t-2} \left(\frac{u_{t-1}^2}{1 + e^{\gamma u_{t-1} \sqrt{h_{t-1}}}} \right).$$

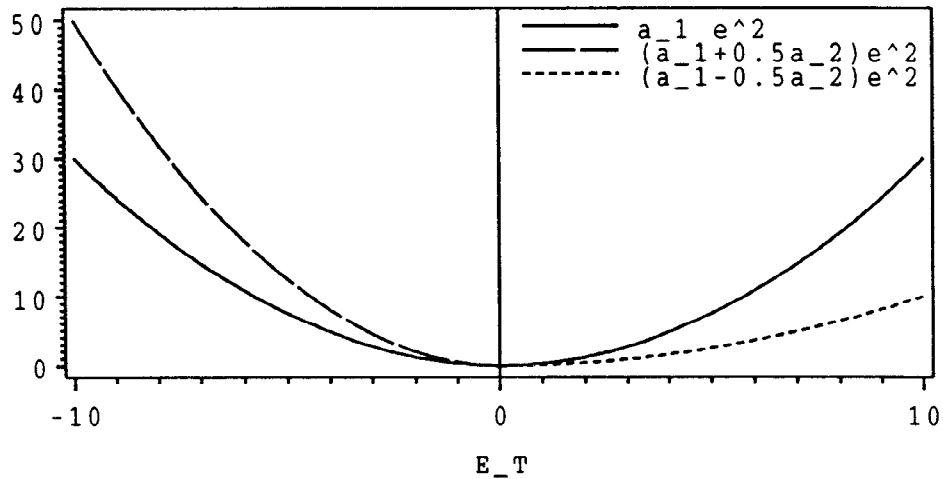


Figure 3

The upper and lower boundaries of a news-impact curve for an ST-GARCH (1,1,1) model.

For the upper regime in Equation 4,

$$E(\varepsilon_t^4) = \frac{3\omega^2(1 + \alpha_1 + 0.5\alpha_2 + \beta)}{(1 - \alpha_1 - 0.5\alpha_2 - \beta)(1 - \beta^2 - 2(\alpha_1 + 0.5\alpha_2)\beta - 3(\alpha_1 + 0.5\alpha_2)^2)}.$$

The news-impact curve for the ST-GARCH(1,1,1) model is:

$$h_t = K + \varepsilon_{t-1}^2(\alpha_1 + \alpha_2 F(\varepsilon_{t-1}, \gamma)).$$

Figure 3 pictures the upper and lower boundaries corresponding to an ST-GARCH(1,1,1) model.

3 Testing

There are several ways to test for asymmetry in volatility in an ST-GARCH model. In Equation 1 there is no asymmetry when $H_{01} : \gamma = 0$ or when $H_{02} : \alpha_{2i} = 0 \forall i$. If H_{01} is true, then α_{2i} can take any value; or, when H_{02} is true, γ can take any value. In other words, the model is unidentified under the null hypothesis.

I study the behavior of an LM test for the two null hypotheses. Under H_{01} , I derive an LM test that has a standard distribution, while under H_{02} the LM test has a nonstandard distribution.

3.1 The LM test with a standard distribution

Consider an ST-GARCH($p, 0, d$) model,

$$h_t = \omega + \sum_{i=1}^p \alpha_{1i} \varepsilon_{t-i}^2 + \left(\sum_{i=1}^p \alpha_{2i} \varepsilon_{t-i}^2 \right) F(\varepsilon_{t-d}, \gamma), \quad (5)$$

for which the hypothesis of interest is $H_0 : \gamma = 0$, $H_1 : \gamma > 0$. Under conditional normality, the log-likelihood of observation t is:

$$l_t = -\frac{1}{2} \log h_t - \frac{1}{2} \frac{\varepsilon_t^2}{h_t}.$$

Under the null hypothesis, the parameter vector $\alpha_2 = (\alpha_{21}, \alpha_{22}, \dots, \alpha_{2p})'$ is unidentified. Following Davies (1977, 1987), I keep α_2 fixed. The parameter vector for which the score is calculated is $\theta \equiv (\omega \ \alpha_{11} \dots \alpha_{1p} \ \gamma)'$, and the score vector under the null hypothesis is:

$$S(\theta, \alpha_2)_{|H_0} = \frac{1}{2} \sum_t z_t v_t,$$

with

$$v_t = \frac{\varepsilon_t^2}{\tilde{h}_t} - 1, \quad z_t = \left[\frac{1}{\tilde{h}_t} \quad \frac{\varepsilon_{t-1}^2}{\tilde{h}_t} \quad \dots \quad \frac{\varepsilon_{t-p}^2}{\tilde{h}_t} \quad \frac{-\varepsilon_{t-d} \sum_i \alpha_{2i} \varepsilon_{t-i}^2}{4\tilde{h}_t} \right]' = [z'_{1t} \mid z_{2t}(\alpha_2)]',$$

where \tilde{h}_t is the conditional variance under the null, and the partition of the $(p+2) \times 1$ vector z_t is such that $z_{2t}(\alpha)$ is equal to the last element of z_t .

The asymptotic variance of the score under the null hypothesis is:

$$\begin{aligned} V &= E(S(\theta, \alpha_2)S(\theta, \alpha_2)') = \frac{1}{4} E\left(\left(\sum_t z_t v_t\right)\left(\sum_t z_t v_t\right)'\right) \\ &= \frac{1}{2} E\left(\sum_t z_t z_t'\right), \end{aligned}$$

where the last equality follows because $S(\theta, \alpha_2)$ is a Martingale difference, and under conditional normality $E(v_t^2) = 2$. V can be consistently estimated by:

$$\hat{V} = \frac{1}{2} \left(\frac{\sum_t \hat{z}_t \hat{z}_t'}{T} \right),$$

where T is the sample size.

A general form of the LM statistic for $H_0 : \gamma = 0$ is:

$$\begin{aligned} \text{LM}_\gamma(\alpha_2) &= \frac{1}{2} \left(\sum_t \hat{v}_t \hat{z}_t' \right) \left(\sum_t \hat{z}_t \hat{z}_t' \right)^{-1} \left(\sum_t \hat{z}_t \hat{v}_t \right) \\ &= \frac{1}{2} \left(\sum_t \hat{v}_t \hat{z}_{2t} \right)^2 \left\{ \sum_t \hat{z}_{2t}^2 - \sum_t \hat{z}_{2t} \hat{z}'_{1t} \left(\sum_t \hat{z}_{1t} \hat{z}'_{1t} \right)^{-1} \sum_t \hat{z}_{1t} \hat{z}_{2t} \right\}^{-1}, \end{aligned} \quad (6)$$

where \hat{v}_t and \hat{z}_t are evaluated at the maximum-likelihood estimates under the null hypothesis.

Asymptotically, an equivalent test (Harvey 1990) to Equation 6 can be constructed using the following auxiliary regression:

$$\hat{v}_t = \hat{z}'_{1t} \beta_1 + \hat{z}_{2t}(\alpha_2) \beta_2 + u_t. \quad (7)$$

The test is:

$$\widetilde{\text{LM}}_\gamma(\alpha_2) = TR^2 = T \frac{SSR_0 - SSR(\alpha_2)}{SSR_0},$$

where R^2 is the multiple coefficient of determination of the auxiliary regression in Equation 7, SSR_0 is the sum of the squared residuals in Equation 7 under the null $H_0 : \gamma = 0$ ($\beta_2 = 0$), and $SSR(\alpha_2)$ is the sum of the squared residuals under the alternative.

If α_2 were known, the test would be distributed as a χ^2 with one degree of freedom, but it is the dependence on the unknown parameter α_2 that in general may make the test not behave in the standard fashion. Davies (1977) proposed the following test:

$$\text{LM}_\gamma = \sup_{\alpha_2} \widetilde{\text{LM}}_\gamma(\alpha_2) = T \frac{SSR_0 - \inf_{\alpha_2} SSR(\alpha_2)}{SSR_0}, \quad (8)$$

which in general has an unknown distribution under the null hypothesis. However, in the present case, the test of Equation 6 or the equivalent test of Equation 8 has a χ^2 -distribution. This standard behavior is seen from the following auxiliary regression:

$$\hat{v}_t = \hat{z}'_{1t} \beta_1 + \beta_2 \sum_i \left(-\alpha_{2i} \frac{\varepsilon_{t-d} \varepsilon_{t-i}^2}{4} \frac{1}{\tilde{h}_t} \right) + u_t = \hat{z}'_{1t} \beta_1 - \sum_i \phi_i \left(\frac{\varepsilon_{t-d} \varepsilon_{t-i}^2}{4} \frac{1}{\tilde{h}_t} \right) + u_t, \quad (9)$$

Table 1

Size of the LM test

Model under H_0 GARCH(1,1)	Sample Size	Nominal Size			χ_1^2	
		10%	5%	1%	Mean	Variance
$\omega = 0.1, \alpha = 0.1,$ $\beta = 0.8$	500	10.20	4.55	0.08	0.98	1.80
	1,500	11.45	5.50	1.10	1.03	2.09
$\omega = 0.02, \alpha = 0.03,$ $\beta = 0.95$	500	9.35	4.25	0.07	0.94	1.83
	1,500	10.75	5.25	1.10	1.02	1.98

Table 2

Power of the LM test

Model under H_1 ST-GARCH(1,1,1)	Sample Size	Rejection Frequencies (%)		
		Nominal Size		
		10%	5%	1%
$\omega = 0.1, \alpha_1 = 0.1,$ $\alpha_2 = 0.1, \beta = 0.8, \gamma = 1$	500	32.35	21.95	7.05
	1,500	70.00	57.60	34.10
$\omega = 0.02, \alpha_1 = 0.03,$ $\alpha_2 = 0.05, \beta = 0.95, \gamma = 20$	500	31.95	21.00	7.85
	1,500	78.55	67.95	45.95

where:

$$\phi_i = \beta_2 \alpha_{2i}, \quad i = 1 \dots p. \quad (10)$$

If the Equation-10 restriction holds, the auxiliary regressions in Equations 7 and 9 are identical, and the \inf_{α_2} SSR is achieved when the least-squares estimation of Equation 9 is performed. Consequently, under the null hypothesis $H_0 : \gamma = 0$, the LM test of Equation 8 is χ^2 -distributed, with p degrees of freedom.

If the model contains a GARCH term, as in the model described by Equation 2, the testing becomes more cumbersome because of the iterative term in the score, but the derived LM test is still distributed as χ^2 . Equation (6) is still valid. For an ST-GARCH(1,1,1) model, the parameter vector is $(\omega \ \alpha_{11} \ \beta \ \gamma)$ with α_{21} fixed, and the vector z_t is equal to:

$$z_t = \left[\frac{1}{\tilde{h}_t} + \frac{\beta}{\tilde{h}_t} \frac{\partial h_{t-1}}{\partial \omega} \quad \frac{\varepsilon_{t-1}^2}{\tilde{h}_t} + \frac{\beta}{\tilde{h}_t} \frac{\partial h_{t-1}}{\partial \alpha_{11}} \quad \frac{\tilde{h}_{t-1}}{\tilde{h}_t} + \frac{\beta}{\tilde{h}_t} \frac{\partial h_{t-1}}{\partial \beta} \quad - \frac{\alpha_{21}}{4} \frac{\varepsilon_{t-1}^3}{\tilde{h}_t} + \frac{\beta}{\tilde{h}_t} \frac{\partial h_{t-1}}{\partial \gamma} \right]'$$

In this case, the test LM_γ in Equation 8 is distributed as a χ^2 , with one degree of freedom.

If the normality assumption is not satisfied, the LM test can be made robust to departures from normality, as Bollerslev and Wooldridge have shown (1992).²

I perform a Monte Carlo experiment to assess the finite sample properties of the Equation-6 test. Tables 1 and 2 display the size and power of the test. The experiment consists of 2,000 replications.

In both tables, the parameters of the models have been chosen according to the estimated values reported in Section 4. I have chosen two sample sizes, 500 and 1,500 observations. The empirical size of the test is close to the nominal size for both samples. In the last two columns, I report the empirical mean and the empirical variance of the test that is consistent with the theoretical mean ($= 1$) and variance ($= 2$) of a χ^2 with one degree of freedom. The test is also powerful: for 1,500 observations and 5% nominal size, the test rejects the null hypothesis in 58% of the cases for a small $\gamma = 1$, and in 68% of the cases for a large $\gamma = 20$.

3.2 The LM test with nonstandard distribution

Consider the ST-GARCH($p, 0, d$) model of Equation 5, where the hypothesis of interest is $H_0 : \alpha_2 = 0$. Under this null, the parameter γ is not identifiable. Proceeding as in the previous section, I keep γ fixed. The parameter vector for which the score is calculated is $\theta \equiv (\omega \ \alpha_{11} \dots \alpha_{1p} \ \alpha_{21} \dots \alpha_{2p})'$. The score vector under

²Consider the vector θ partitioned as $(\theta_1 \ \gamma)$. The asymptotic variance-covariance matrix of the score for a general assumption on the conditional distribution function and under the null hypothesis is:

$$W_{\gamma\gamma} \equiv V_{\gamma\gamma} - H_{\gamma\theta_1} H_{\theta_1\theta_1}^{-1} V_{\theta_1\gamma} - V_{\gamma\theta_1} H_{\theta_1\theta_1}^{-1} H_{\theta_1\gamma} + H_{\gamma\theta_1} H_{\theta_1\theta_1}^{-1} V_{\theta_1\theta_1} H_{\theta_1\theta_1}^{-1} H_{\theta_1\gamma},$$

where H is the Hessian matrix, and V is the outer product of the score.

the null hypothesis is:

$$S(\theta, \gamma)_{|H_0} = \frac{1}{2} \sum_t z_t v_t,$$

with

$$v_t = \frac{\varepsilon_t^2}{\tilde{h}_t} - 1$$

$$z_t = \left[\frac{1}{\tilde{h}_t} \frac{\varepsilon_{t-1}^2}{\tilde{h}_t} \dots \frac{\varepsilon_{t-p}^2}{\tilde{h}_t} \frac{\varepsilon_{t-1}^2 F(\varepsilon_{t-d}, \gamma)}{\tilde{h}_t} \dots \frac{\varepsilon_{t-p}^2 F(\varepsilon_{t-d}, \gamma)}{\tilde{h}_t} \right]' = [z'_{1t} | z_{2t}(\gamma)]',$$

where \tilde{h}_t is the conditional variance under the null hypothesis, and the partition of the vector z_t is such that $z_{2t}(\gamma)$ is equal to those elements of z_t affected by the function F . Under conditional normality, the asymptotic variance of the score under the null hypothesis is:

$$V = E(S(\theta, \gamma)S(\theta, \gamma)') = \frac{1}{2} E \left(\sum_t z_t z_t' \right).$$

A general form of the LM statistic for $H_0 : \alpha_2 = 0$ is:

$$LM_{\alpha_2}(\gamma) = \frac{1}{2} \left(\sum_t \hat{v}_t \hat{z}'_{2t} \right) \left\{ \sum_t \hat{z}_{2t} \hat{z}'_{2t} - \sum_t \hat{z}_{2t} \hat{z}'_{1t} \left(\sum_t \hat{z}_{1t} \hat{z}'_{1t} \right)^{-1} \sum_t \hat{z}_{1t} \hat{z}'_{2t} \right\}^{-1} \left(\sum_t \hat{v}_t \hat{z}_{2t} \right), \quad (11)$$

where \hat{v}_t and \hat{z}_t are evaluated at the maximum-likelihood estimates under the null hypothesis.

If γ were known, the LM test of Equation 11 would be distributed as a χ^2 with p degrees of freedom. In general, γ is not known, and unfortunately in this case, the dependence on γ cannot be overcome as it can in the case of the LM test in Equation 6. This is because the dependence of $z_{2t}(\gamma)$ on γ is nonlinear. The conventional maximum-likelihood theory does not apply when testing $H_0 : \alpha_2 = 0$. Nevertheless, testing is still possible. Davies (1977) suggested the following test:

$$LM_{\alpha_2} = \sup_{\gamma} LM_{\alpha_2}(\gamma), \quad (12)$$

for which the probability distribution is unknown under the null hypothesis. However, Davies (1977) provided a bound for tests of the type shown by Equation 12. Other alternative testing procedures can be constructed, as discussed by Hansen (1996), who in a regression framework proposed finding the distribution of Equation 12 via simulation. Lukkonen, Saikkonen, and Teräsvirta (1988) proposed a battery of LM tests where the function F was linearized with a Taylor expansion to break the nonlinear dependence on the parameter γ . Hagerud (1997) implemented a second-order Taylor expansion of the transition function around zero, and developed the asymptotic test in Equation 11 for an ST-GARCH(1,1,1) model. It turns out that this test is equivalent to the test in Equation 6 of Section 3.1. Hagerud (1997) also performed further Monte Carlo simulations that showed that the test has very good size and good power against a wide array of alternative models.

4 Applications

In this section, a smooth-transition GARCH model is estimated to financial data. The estimation is performed with a quasi-maximum-likelihood (QML) procedure. Proving consistency and asymptotic normality of the QML estimator for GARCH processes is not a trivial exercise. Results are only available under the assumption of conditional normality, and only for a limited class of processes, mainly GARCH(1,1) (see Lumsdaine 1996; Lee and Hansen 1994) and ARCH(p) (see Weiss 1986). Consistency and asymptotic normality of the QML estimator for the ST-GARCH model may follow along the lines developed by Lumsdaine (1996). This exercise is beyond the scope of this article and I defer it to future research.

Table 3

Estimation of conditional variance. Exchange rates*

$$h_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-1}^2 F(\varepsilon_{t-1}, \gamma) + \beta h_{t-1}$$

$$h_t = \omega + \alpha (|\varepsilon_{t-1}| - \delta \varepsilon_{t-1})^2 + \beta h_{t-1}$$

	British Pound		Deutsche Mark	
	Smooth	Threshold	Smooth	Threshold
c^a	-0.057	-0.058	-0.062	-0.053
(t test) ^b	(-3.2)	(-3.3)	(-3.3)	(-3.4)
(Robust t)	(-3.3)	(-3.4)	(-3.7)	(-3.0)
ω	0.007 (2.8) (1.7)	0.007 (2.9) (1.7)	0.010 (2.3) (1.9)	0.009 (2.9) (1.9)
α_1 α	0.037 (3.9) (3.1)	0.035 (3.6) (2.5)	0.099 (5.1) (4.9)	0.086 (5.7) (4.9)
α_2	0.036 (2.6) (1.5)	— — —	0.135 (0.8) (0.7)	— — —
γ δ	14.269 (0.0) (0.0)	0.273 (2.5) (1.2)	0.862 (0.6) (0.5)	0.153 (2.8) (1.6)
β	0.943 (68.3) (49.4)	0.942 (70.4) (49.6)	0.886 (40.6) (37.1)	0.895 (54.0) (41.3)
Log likelihood	-1196.8	-1196.8	-1258.6	-1259.7
Skewness ^c	-0.31	-0.31	0.17	0.18
Kurtosis ^c	4.43	4.42	3.61	3.65
Bera-Jarque ^c	124.3	124.3	25.2	29.6
Q(12) ^d	7.6	7.8	10.2	10.2
Robust LM ^e	1.00	2.43	5.96	4.51
p -value	0.317	0.119	0.014	0.033

*See notes at the end of Table 4.

I estimate smooth-transition and threshold GARCH models to exchange rates and stock returns. The first data set consists of returns to four exchange rates: the British pound (BP), the Deutsche mark (DM), the Italian lira (LI), and the Swiss franc (SF).³ The data are opening bid prices of the foreign currency against the U.S. dollar in the New York Foreign Exchange Market from March 1, 1980 to January 28, 1985, for a total of 1,245 daily observations. The second data set consists of 7,420 daily returns to the Standard & Poor 500 (S&P 500) index from July 2, 1962 to December 31, 1991, and 1,330 weekly returns to individual stocks—IBM, NCR, and Unisys—from July 1962 to December 1987. These data have been extracted from the Center for Research on Stock Prices (CRSP) tapes.

Table 3 contains the estimation results for the exchange rates. For every currency, I estimate and test a smooth-transition GARCH model (ST) and a threshold model (TH). Since departures from conditional normality are present (mainly due to leptokurtosis and a mild skewness), I present robust t -statistics for the coefficient estimates and robust LM statistics of the form of Equation 6 to test for the existence of a smooth-transition mechanism or threshold. With the exception of the BP, asymmetries in variance are present for the other three currencies. For the BP, the robust LM tests for $H_0 : \gamma = 0$ and $H_0 : \delta = 0$ fail to reject their respective null hypothesis at the 5% significance level.

To decide if asymmetries are better modeled by an ST or a TH model, the following considerations should be kept in mind. First, in a smooth-transition model, the t -statistic corresponding to α_2 does not have the standard Student's t distribution for $H_0 : \alpha_2 = 0$, because the parameter γ is not identified under the null hypothesis. Second, the standard errors corresponding to the transition parameter γ are big, rendering small t -statistics. This problem is more severe when γ is large. This is due to the structure of the model, mainly to

³This data has been kindly provided by Richard T. Baillie.

Table 3 (cont.)

Estimation of conditional variance. Exchange rates*

$$h_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-1}^2 F(\varepsilon_{t-1}, \gamma) + \beta h_{t-1}$$

$$h_t = \omega + \alpha (|\varepsilon_{t-1}| - \delta \varepsilon_{t-1})^2 + \beta h_{t-1}$$

	Italian Lira		Swiss Franc	
	Smooth	Threshold	Smooth	Threshold
c^a	-0.082	-0.081	-0.059	-0.058
(t test) ^b	(-5.0)	(-5.0)	(-2.9)	(-2.9)
(Robust t)	(-5.4)	(-5.3)	(-3.2)	(-3.1)
ω	0.013 (3.1)	0.013 (3.2)	0.005 (1.6)	0.007 (1.9)
α_1 α	(2.7)	(2.8)	(1.3)	(1.8)
	0.137 (6.2)	0.119 (6.0)	0.074 (4.9)	0.070 (4.9)
	(5.2)	(5.2)	(4.5)	(4.7)
α_2	0.189 (1.2)	— (1.2)	0.174 (0.3)	— (0.3)
	(1.2)	—	(0.2)	—
γ δ	0.980 (0.9)	0.183 (3.8)	0.404 (0.3)	0.161 (2.3)
	(0.7)	(1.9)	(0.2)	(1.8)
β	0.839 (33.6)	0.850 (35.3)	0.919 (55.9)	0.917 (55.4)
	(29.4)	(33.0)	(51.5)	(53.0)
Log likelihood	-1130.7	-1131.7	-1350.3	-1351.8
Skewness ^c	0.07	0.08	0.14	0.15
Kurtosis ^c	3.98	4.00	3.35	3.37
Bera-Jarque ^c	50.7	54.3	10.4	12.1
Q(12) ^d	8.9	9.0	9.1	9.2
Robust LM ^e	6.41	5.46	4.84	3.10
p -value	0.011	0.019	0.027	0.078

* See notes at the end of Table 4.

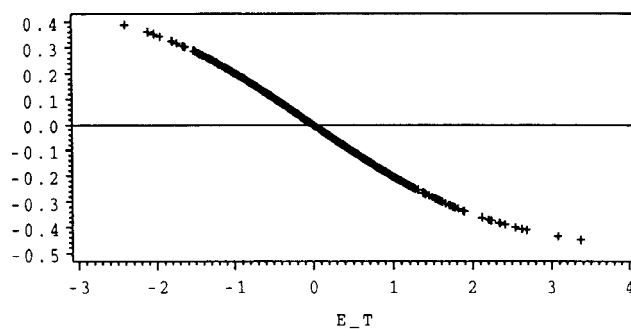
the behavior of the function $F(\varepsilon_{t-1}, \gamma)$. In an ST-GARCH(1,1,1) model,

$$\frac{\partial h_t}{\partial \gamma} = \alpha_2 \varepsilon_{t-1}^2 F_\gamma = -\alpha_2 \frac{\varepsilon_{t-1}^3 e^{\gamma \varepsilon_{t-1}}}{(1 + e^{\gamma \varepsilon_{t-1}})^2}.$$

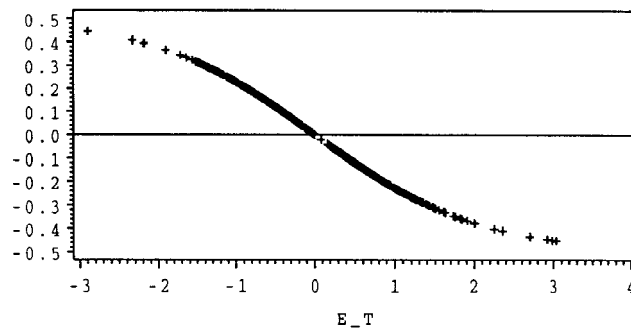
This partial derivative is part of the score. In calculating the outer product of the score and the hessian, the above element is squared, producing numbers that are close to zero, particularly for large γ . This effect is reflected in a large variance for $\hat{\gamma}$. Consequently, on testing for the existence of a smooth-transition mechanism, I rely on the LM test in Equation 6 that is explained in the previous section.

If a threshold model is to be fitted to the DM, LI, and SF, the robust t -statistic for $H_0 : \delta = 0$ fails to reject the null hypothesis at the conventional significance levels. When a robust LM test is performed for H_0 , I fail to reject the null at the 5% significance level for the SF, and I reject the null for the DM (p -value = 3.3%) and for the LI (p -value = 1.9%). The estimation of a smooth-transition mechanism in variance offers different results. The robust LM test in Equation 6 for $H_0 : \gamma = 0$ rejects the null hypothesis for the DM, LI, and SF currencies. Furthermore, comparing the LM tests for both models, the ST model offers larger values of the test than the TH model. The strength of the rejection (smaller p -values of the LM tests) points toward a preference for smooth-transition models. In fact, the values of the transition parameter γ are consistent with the preference for ST models: these are 0.86 for the DM, 0.98 for the LI, and 0.40 for the SF. Figure 4 shows the transition function $F(\varepsilon_{t-1}, \gamma)$ for these currencies. It is interesting to note that the condition $\alpha_2 = 4\alpha\delta$ for convergence of the ST model to the TH model is not satisfied. For the DM, $4\alpha\delta = 0.053 < 0.135 = \alpha_2$; for the LI, $4\alpha\delta = 0.088 < 0.189 = \alpha_2$; and for the SF, $4\alpha\delta = 0.045 < 0.174 = \alpha_2$.

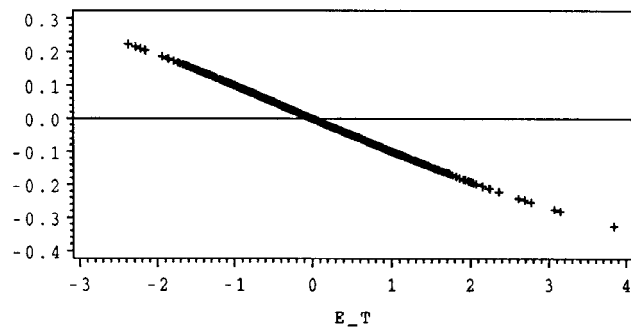
Table 5 displays the limiting processes of the four currencies. During periods of good news (lower regime, $\varepsilon_t > 0$), the markets exhibit little time-varying volatility; in particular, for the SF, the lower regime is characterized by a constant variance. However, when a negative shock hits the market (upper regime), the markets become highly volatile and the conditional variance is characterized by an integrated GARCH(1,1) process.



(a)



(b)

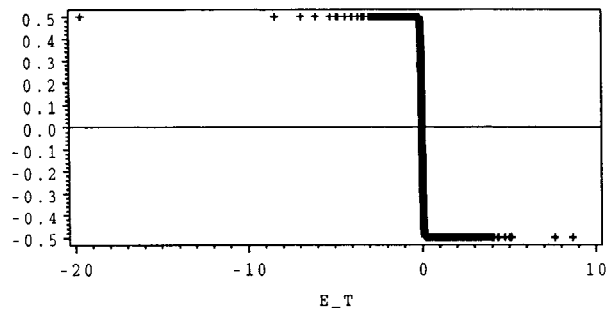


(c)

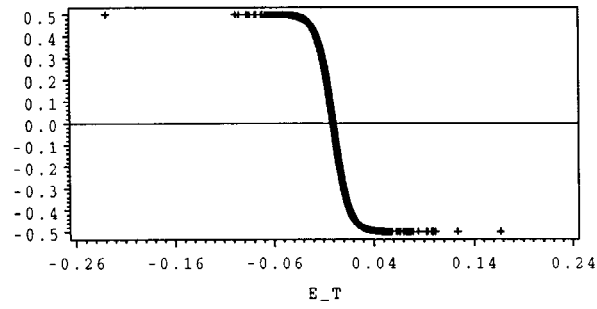
Figure 4

The empirical transition function $F(\varepsilon_{t-1}, \gamma)$ for the Deutsche mark (a); the Italian lira (b); and the Swiss franc (c) exchange rates.

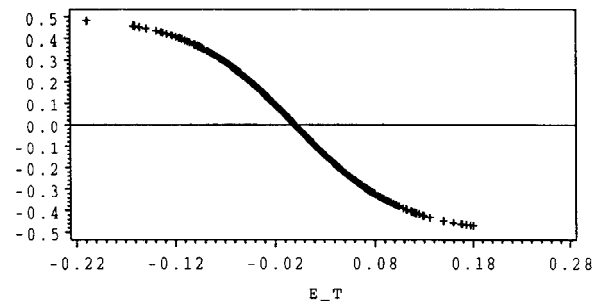
Table 4 contains the estimation results for stock returns. For three individual stocks and for the S&P 500 index, asymmetries in variance are strong. In the threshold model, the robust t -statistics for $H_0 : \delta = 0$ reject the null hypothesis at the conventional levels, except for NCR, in which the robust t -test is 1.6. On the other hand, the robust LM test for $H_0 : \delta = 0$ strongly rejects the null for the three stocks and the index, with p -values much smaller than 1%. If a smooth-transition mechanism is fitted, the robust LM test of Equation 6 for $H_0 : \gamma = 0$ rejects the null hypothesis, but with larger p -values than the threshold model. The strength of the rejection points toward a preference for a threshold model in stock returns. In fact, the values of the transition parameter γ are large, ranging from 133.58 for IBM to 18.88 for Unisys. Figure 5 presents the transition function $F(\varepsilon_{t-1}, \gamma)$ for these stocks and the S&P 500 index. For practical purposes, these plots show that indeed, a threshold model is present.



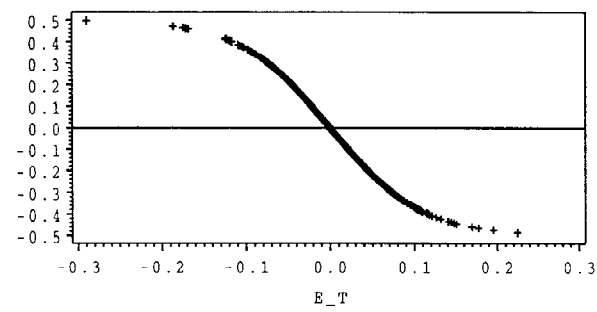
(a)



(b)



(c)



(d)

Figure 5

The empirical transition function $F(\varepsilon_{t-1}, \gamma)$ for the Standard & Poor 500 index (a); IBM stock (b); NCR stock (c); and Unisys stock (d).

Table 4

Estimation of conditional variance. Stock returns

$$b_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-1}^2 F(\varepsilon_{t-1}, \gamma) + \beta b_{t-1}$$

$$b_t = \omega + \alpha (|\varepsilon_{t-1}| - \delta \varepsilon_{t-1})^2 + \beta b_{t-1}$$

	S&P 500		IBM	
	Smooth	Threshold	Smooth	Threshold
c^d	0.026	0.026	0.002	0.002
(t -test) ^b	(2.9)	(3.0)	(3.1)	(3.0)
(Robust t)	(2.8)	(2.9)	(3.0)	(3.0)
MA(1)	0.170	0.172	—	—
	(14.3)	(14.5)	—	—
	(13.9)	(14.1)	—	—
ω	0.003	0.003	0.000	0.000
	(5.0)	(5.1)	(2.0)	(2.5)
	(3.8)	(4.0)	(1.4)	(2.0)
α_1 α	0.071	0.062	0.050	0.043
	(25.6)	(18.8)	(5.5)	(3.9)
	(7.0)	(6.8)	(4.1)	(2.6)
α_2	0.082	—	0.076	—
	(16.7)	—	(4.9)	—
	(4.9)	—	(3.1)	—
γ δ	39.43	0.318	133.58	0.498
	(0.0)	(10.1)	(0.3)	(3.9)
	(0.0)	(6.0)	(0.2)	(2.0)
β	0.929	0.931	0.943	0.933
	(329.2)	(339.3)	(90.6)	(70.3)
	(104.3)	(105.5)	(58.3)	(50.3)
Log likelihood	-8379.3	-8378.9	2845.3	2845.8
Skewness ^c	-0.29	-0.29	0.00	0.00
Kurtosis ^c	6.50	6.40	5.38	5.41
Bera-Jarque ^c	3890.7	3885.5	313.6	321.7
Q(12) ^d	11.1	10.9	19.8	19.9
Robust LM ^e	5.29	37.19	5.76	9.21
p -value	0.0214	1.07E-09	0.016	0.002

The limiting processes of these series are shown in Table 5. The lower regime is characterized by almost no time-varying volatility. In fact, for NCR and Unisys, the lower regime has a constant variance. However, in the upper regime for all stocks and the S&P 500 index, the conditional variance is characterized by an integrated GARCH(1,1) model.

5 Conclusions

The empirical literature on asymmetry in variance, best known as the leverage effect, has focused on models with threshold with applications to stock returns. In this paper, I have introduced a smooth-transition GARCH model (ST-GARCH). The asymmetric response of conditional variances to positive and negative news has been modeled with a smooth-transition mechanism. The ST-GARCH model permits the existence of intermediate regimes between high-volatility and low-volatility regimes. In this sense, it is a more complete specification that also nests the threshold model, since for certain parameter values the smooth transition collapses to the threshold model. An important characteristic of the ST-GARCH model is it allows for an on-off ARCH effect. The switching between regimes is constrained by an upper regime and a lower regime. Under certain conditions, the lower regime may be a process with a constant variance.

Testing for the existence of a smooth-transition mechanism presents some difficulties, since under the null hypothesis there are nuisance parameters that are not identified. I have presented two types of LM tests. If the null hypothesis involves the smoothness parameter, $H_0 : \gamma = 0$, an LM test can be constructed that is χ^2 -distributed with p degrees of freedom. If the identification problem had not existed, the LM test would have been χ^2 -distributed with one degree of freedom. A Monte Carlo simulation shows that the test has very good size and good power. If the null hypothesis involves the parameter vector α_2 , $H_0 : \alpha_2 = 0$, the corresponding LM test is not χ^2 -distributed, and simulation techniques or a Taylor expansion of the transition function must be used to assess its probability density function.

Table 4 (cont.)

Estimation of conditional variance. Stock returns

$$h_t = \omega + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-1}^2 F(\varepsilon_{t-1}, \gamma) + \beta h_{t-1}$$

$$h_t = \omega + \alpha (|\varepsilon_{t-1}| - \delta \varepsilon_{t-1})^2 + \beta h_{t-1}$$

	NCR		Unisys	
	Smooth	Threshold	Smooth	Threshold
c^d	0.002	0.002	0.002	0.000
(t test) ^b	(2.5)	(2.2)	(2.1)	(1.7)
(Robust t)	(2.4)	(2.1)	(2.1)	(0.3)
ω	0.000	0.000	0.000	0.000
	(2.1)	(2.5)	(3.1)	(3.3)
	(1.9)	(2.3)	(2.2)	(2.1)
α_1 α	0.037	0.032	0.073	0.063
	(4.7)	(2.9)	(5.2)	(4.6)
	(3.1)	(1.9)	(2.8)	(2.5)
α_2	0.087	—	0.143	—
	(2.1)	—	(1.5)	—
	(1.3)	—	(1.4)	—
γ δ	19.25	0.538	18.88	0.409
	(1.1)	(2.4)	(0.8)	(3.7)
	(0.7)	(1.6)	(0.8)	(2.3)
β	0.953	0.942	0.879	0.877
	(87.5)	(70.1)	(34.8)	(35.9)
	(66.3)	(54.4)	(21.2)	(19.8)
Log likelihood	2258.4	2258.2	2270.5	2269.0
Skewness ^c	0.12	0.12	0.00	0.04
Kurtosis ^c	3.82	3.79	4.63	4.65
Bera-Jarque ^c	40.4	38.2	147.1	151.2
Q(12) ^d	19.1	19.5	9.20	9.60
Robust LM ^e	14.29	17.67	6.08	7.31
p -value	.00015	.00002	0.0136	0.0068

^a c is the constant in the mean equation.^b t statistics are in parenthesis. The first number is a nonrobust t test, and the second is robust to departure from conditional normality of the standardized residuals. The t statistics corresponding to α_2 are not distributed as Student's t . Their distribution is unknown under $H_0 : \alpha_2 = 0$.^c Skewness and kurtosis coefficients of the standardized residuals $\varepsilon_t/\sqrt{h_t}$. Bera-Jarque test for normality of $\varepsilon_t/\sqrt{h_t}$, distributed as a χ^2_2 .^d Q(12) is the Box-Pierce statistic for serial correlation in $\varepsilon_t/\sqrt{h_t}$.^e In the smooth-transition model, the robust LM test is for $H_0 : \gamma = 0$. In the threshold model, the robust LM test is for $H_0 : \delta = 0$. In both models, the robust LM test is χ^2_1 distributed.**Table 5**Lower ($F = -1/2$) and upper ($F = 1/2$) regimes

	Lower Regime		Upper Regime	
	$\alpha_1 - \frac{1}{2}\alpha_2$	Persistence	$\alpha_1 + \frac{1}{2}\alpha_2$	Persistence
British Pound	0.02	0.96	0.06	1.00
Deutsche Mark	0.03	0.89	0.17	1.03
Italian Lira	0.04	0.88	0.20	1.04
Swiss Franc	0.00	—	0.16	1.08
S&P 500	0.03	0.96	0.11	1.04
IBM	0.01	0.95	0.09	1.03
NCR	0.00	—	0.08	1.03
Unisys	0.00	—	0.14	1.02

I have applied the estimation and testing of an ST-GARCH model to exchange rates and stock returns and compared the likelihood of an ST-GARCH model versus a threshold model. In agreement with the present literature, stock returns are better modeled with a threshold model. If a threshold model is estimated for exchange rates, it will be easy to conclude that there is no asymmetric response of the variance in exchange rates. However, it turns out that exchange rates are better modeled with a smooth-transition specification. The transition function is very smooth, with the smoothness parameter between 0.4 and 1.0. Furthermore, the on-off ARCH effect is present in some of the stocks and exchange rates.

References

- Andrews, D. W. K., and W. Ploberger (1994). "Optimal tests when a nuisance parameter is present only under the alternative." *Econometrica*, 62(6): 1383–1414.
- Black, F. (1976). "Studies in stock price volatility changes." *Proceedings of the American Statistical Association, Business and Economics Statistics Section*. American Statistical Association, pp. 177–181.
- Bollerslev, T. (1986). "Generalized autoregressive conditional heteroskedasticity." *Journal of Econometrics*, 31: 307–327.
- Bollerslev, T., and J. M. Wooldridge (1992). "Quasi-maximum likelihood estimation and inference in dynamic models with time-varying covariances." *Econometrics Reviews*, 11: 143–172.
- Cai, J. (1994). "A Markov model of switching-regime ARCH." *Journal of Business and Economic Statistics*, 12: 309–316.
- Christie, A. (1982). "The stochastic behavior of common stock variance: Value, leverage, and interest rate effects." *Journal of Financial Economics*, 10: 407–432.
- Davies, R. B. (1977). "Hypothesis testing when a nuisance parameter is present only under the alternative." *Biometrika*, 64: 247–254.
- Davies, R. B. (1987). "Hypothesis testing when a nuisance parameter is present only under the alternative." *Biometrika*, 74: 33–43.
- Ding, Z., C. W. J. Granger, and R. F. Engle (1993). "A long memory property of stock market returns and a new model." *Journal of Empirical Finance*, 1: 83–106.
- Dueker, M. J. (1997). "Markov switching in GARCH processes and mean-reverting stock market volatility." *Journal of Business and Economic Statistics*, 12: 309–316.
- Engle, R. F. (1982). "Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation." *Econometrica*, 50: 987–1008.
- Engle, R. F. (1990). "Discussion on Schwert (1990)." *Review of Financial Studies*, 3: 103–106.
- Engle, R. F., and V. K. Ng (1993). "Measuring and testing the impact of news on volatility." *Journal of Finance*, 48: 1749–1778.
- Glosten, L., R. Jagannathan, and D. Runkle (1993). "Relationship between the expected value and the volatility of the nominal excess return on stocks." *Journal of Finance*, 48: 1779–1801.
- Hagerud, G. E. (1997). "A new non-linear GARCH model." PhD Dissertation, Stockholm School of Economics.
- Hamilton, J. D., and R. Susmel (1994). "Autoregressive conditional heteroskedasticity and changes in regime." *Journal of Econometrics*, 64: 307–333.
- Hansen, B. C. (1996). "Inference when a nuisance parameter is not identified under the null hypothesis." *Econometrica*, 64: 413–430.
- Harvey, A. C. (1990). *The Econometric Analysis of Time Series*. New York: Philip Allan.
- Lee, S. W., and B. E. Hansen (1994). "Asymptotic theory for the GARCH(1,1) quasi-maximum likelihood estimator." *Econometric Theory*, 10: 29–52.
- Li, C. W., and W. K. Li (1996). "On a double-threshold autoregressive heteroskedastic time series model." *Journal of Applied Econometrics*, 11: 253–274.
- Liu, J., C. W. Li, and W. K. Li (1997). "On a threshold autoregression with conditional heteroskedastic variances." *Journal of Statistical Planning and Inference*, 62: 279–300.
- Lukkonen, R., P. Saikkonen, and T. Teräsvirta (1988). "Testing linearity against smooth transition autoregressive models." *Biometrika*, 75: 491–499.
- Lumsdaine, R. L. (1996). "Consistency and asymptotic normality of the quasi-maximum likelihood estimator in IGARCH(1,1) and covariance stationary GARCH(1,1) models." *Econometrica*, 64: 575–596.
- Milhøj, A. (1985). "The moment structure of ARCH processes." *Scandinavian Journal of Statistics*, 12: 281–292.
- Nelson, D. (1990). "Conditional heteroskedasticity in asset returns: A new approach." *Econometrica*, 59: 347–370.
- Schwert, G. W. (1990). "Stock volatility and the crash of '87." *Review of Financial Studies*, 3: 77–102.

- Teräsvirta, T. (1994). "Specification, estimation and evaluation of smooth transition autoregressive models." *Journal of the American Statistical Association*, 89: 208–218.
- Weiss, A. A. (1986). "Asymptotic theory for ARCH models: Estimation and testing." *Econometric Theory*, 2: 107–131.
- Zakoian, J. (1994). "Threshold heteroskedastic models." *Journal of Economic Dynamics and Control*, 18: 931–955.

Advisory Panel

Jess Benhabib, New York University
William A. Brock, University of Wisconsin-Madison
Jean-Michel Grandmont, CREST-CNRS—France
Jose Scheinkman, University of Chicago
Halbert White, University of California-San Diego

Editorial Board

Bruce Mizrach (editor), Rutgers University
Michele Boldrin, University of Carlos III
Tim Bollerslev, University of Virginia
Carl Chiarella, University of Technology-Sydney
W. Davis Dechert, University of Houston
Paul De Grauwe, KU Leuven
David A. Hsieh, Duke University
Kenneth F. Kroner, BZW Barclays Global Investors
Blake LeBaron, University of Wisconsin-Madison
Stefan Mittnik, University of Kiel
Luigi Montrucchio, University of Turin
Kazuo Nishimura, Kyoto University
James Ramsey, New York University
Pietro Reichlin, Rome University
Timo Terasvirta, Stockholm School of Economics
Ruey Tsay, University of Chicago
Stanley E. Zin, Carnegie-Mellon University

Editorial Policy

The *SNDE* is formed in recognition that advances in statistics and dynamical systems theory may increase our understanding of economic and financial markets. The journal will seek both theoretical and applied papers that characterize and motivate nonlinear phenomena. Researchers will be encouraged to assist replication of empirical results by providing copies of data and programs online. Algorithms and rapid communications will also be published.